

LOCAL HADAMARD WELL-POSEDNESS AND BLOW-UP FOR REACTION-DIFFUSION EQUATIONS WITH NON-LINEAR DYNAMICAL BOUNDARY CONDITIONS

ALESSIO FISCELLA AND ENZO VITILLARO

ABSTRACT. The paper deals with local well-posedness, global existence and blow-up results for reaction-diffusion equations coupled with nonlinear dynamical boundary conditions. The typical problem studied is

$$\begin{cases} u_t - \Delta u = |u|^{p-2} u & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } [0, \infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -|u_t|^{m-2} u_t & \text{on } [0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded open regular domain of \mathbb{R}^n ($n \geq 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $2 \leq p \leq 1 + 2^*/2$, $m > 1$ and $u_0 \in H^1(\Omega)$, $u_0|_{\Gamma_0} = 0$. After showing local well-posedness in the Hadamard sense we give global existence and blow-up results when Γ_0 has positive surface measure. Moreover we discuss the generalization of the above mentioned results to more general problems where the terms $|u|^{p-2}u$ and $|u_t|^{m-2}u_t$ are respectively replaced by $f(x, u)$ and $Q(t, x, u_t)$ under suitable assumptions on them.

1. INTRODUCTION AND MAIN RESULTS

We consider the problem

$$(1) \quad \begin{cases} u_t - \Delta u = f(x, u) & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } [0, \infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -Q(t, x, u_t) & \text{on } [0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

where $u = u(t, x)$, $t \geq 0$, $x \in \Omega$, $\Delta = \Delta_x$ denotes the Laplacian operator with respect to the x variable, Ω is a bounded open subset of \mathbb{R}^n ($n \geq 1$) of class C^1 (see [9]), with $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with $(n-1)$ -dimensional surface measure σ . These properties of Ω , Γ_0 and Γ_1 will be assumed, without further comments, throughout the paper. The initial datum u_0 belongs to the energy space $H^1(\Omega)$, with the compatibility condition $u_0 = 0$ on Γ_0 . Moreover Q represents a nonlinear dynamical term such that $Q(t, x, v)v \geq 0$, and f represents a nonlinear internal reaction (or source) term, i.e. $f(x, u)u \geq 0$.

When $Q \equiv 0$ problem (1) is an initial-boundary value problem related to a semilinear reaction-diffusion equation with homogeneous Dirichlet – Neumann boundary conditions. In this case local well-posedness, under suitable assumptions on f , can be obtained in a standard way using semigroup theory. See for example [42, 54] or

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[2] combined with [14, Appendix]. There is also a wide literature on global existence and blow-up for such type of problems, starting from the classical paper of Levine [32]. See for example [11, 22, 28, 33, 34, 45], [60, Section 5] and [17, 29, 46, 47, 48]. In this case the concavity method of H. Levine is effective in getting blow-up results.

When $Q(t, x, u_t) = \alpha(t, x)u_t$ problem (1) consists in a reaction-diffusion equation coupled with a linear dynamical boundary condition. For well-posedness results, obtained by semigroup and interpolation theories we refer to [2, 18, 19, 26, 27], while blow-up results were proven in [20, 30]. We also refer to [6] for a physical motivation of dynamical boundary conditions, and to the recent papers [21, 61, 62]. Also in this case the concavity method applies (see [49]) in order to establish blow-up. We also would like to mention the classical local-existence and blow-up results in [35, 36, 38, 39] dealing with the related case when the source f appears on the boundary condition.

When Q is nonlinear but monotone increasing in u_t and, roughly speaking, either $f \equiv 0$ or $-\Delta - f$ is a monotone operator, the existence of global solutions of problem (1) can be proved by applying the results in [16], since the problem can be written as a doubly nonlinear evolution equation in a suitable Banach space. We also refer to [25, 43, 50] for related results. When Q is nonlinear and f appears on the boundary condition instead than in the equation, local and global existence has been studied in [59]. Next, the same boundary condition arises in the literature in connection with the wave equation, i.e. when the heat operator $u_t - \Delta u$ in (1) is replaced by the wave operator $u_{tt} - \Delta u$. In particular we refer to [4, 7, 8, 12, 13, 24, 31, 58]. Finally we would like to mention that our analysis on global behavior of the solutions of (1) is related to the methods in [34]. See also [5, 23, 37, 49].

In this paper we study problem (1) when, roughly, $Q(t, x, u_t) \approx |u_t|^{m-2}u_t$ as $|u_t| \geq 1$, $m > 1$, and $f(x, u) \approx |u|^{p-2}u$, $p \geq 2$, as $|u| \geq 1$. The interest in considering superlinear terms ($m > 2$) is mainly of theoretical nature. However, a physical model involving $Q(t, x, u_t) = u_t + |u_t|^{m-2}u_t$, $m > 2$, is given in Appendix A.

In order to state and prove our results in the simplest possible way we shall first consider the model problem

$$(2) \quad \begin{cases} u_t - \Delta u = |u|^{p-2}u & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } [0, \infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -|u_t|^{m-2}u_t & \text{on } [0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

where $m > 1$, $p \geq 2$. We denote by 2^* the critical exponent of Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, i.e. $2^* = 2n/(n-2)$ when $n \geq 3$ while $2^* = \infty$ when $n = 1, 2$. Moreover we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$, $\|\cdot\|_{q, \Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}$ for $1 \leq q \leq \infty$, and the Hilbert space $H_{\Gamma_0}^1(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$, $\|u\|_{H_{\Gamma_0}^1}^2 := \|u\|_2^2 + \|\nabla u\|_2^2$, where $u|_{\Gamma_0}$ stands for the restriction of the trace of u on $\partial\Omega$ to Γ_0 . The first aim of the paper is to show that problem (2) is well-posed in $H_{\Gamma_0}^1(\Omega)$. The first step in this direction is given by the following result.

Theorem 1. (Local existence and uniqueness) *Let $m > 1$ and*

$$(3) \quad 2 \leq p \leq 1 + \frac{2^*}{2}.$$

Then, given $u_0 \in H_{\Gamma_0}^1(\Omega)$, there is a $T^* = T^*(\|u_0\|_{H_{\Gamma_0}^1}, m, p, \Omega, \Gamma_1) \in (0, 1]$, decreasing in the first variable, such that problem (2) has a unique weak ¹ solution u in $[0, T^*] \times \Omega$. Moreover

$$(4) \quad u \in C([0, T^*]; H_{\Gamma_0}^1(\Omega)),$$

$$(5) \quad u_t \in L^m((0, T^*) \times \Gamma_1) \cap L^2((0, T^*) \times \Omega)$$

and the energy identity

$$(6) \quad \frac{1}{2} \|\nabla u\|_2^2 \Big|_s^t + \int_s^t \|u_t\|_{m, \Gamma_1}^m + \|u_t\|_2^2 = \int_s^t \int_{\Omega} |u|^{p-2} u u_t$$

holds for $0 \leq s \leq t \leq T^*$. Finally

$$(7) \quad \|u\|_{C([0, T^*]; H_{\Gamma_0}^1(\Omega))} \leq 4 \|u_0\|_{H_{\Gamma_0}^1}.$$

Remark 1. The assumption $p \leq 1 + 2^*/2$ in Theorem 1 is quite restrictive when $n \geq 3$, although it appears often in the literature quoted above. Clearly it expresses the assumption that the Nemitski operator $u \mapsto |u|^{p-2}u$ is locally Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Such type of assumptions has been overcome, in the author's knowledge, either by getting additional a-priori estimates, as done for example in [7, 51], or using linear semigroup and interpolation theories, as done for example in [2, 19]. While in this case the nonlinear term Q does not give useful estimates, being active on the boundary, it prevents to use linear theory and interpolation of semigroups. Nonlinear semigroup theory can be used, as in [16], but in this case one still needs to assume that the Nemitski operator above is locally Lipschitz, as in [14]. To prove Theorem 1 we found simpler to first use the monotonicity method of J. L. Lions and then to use a contraction argument.

By using the same energy estimates used to prove Theorem 1 we complete our well-posedness analysis as follows.

Theorem 2. (Continuation and local Hadamard well-posedness) *Under the assumption of Theorem 1, problem (2) has a unique weak maximal solution u in $[0, T_{max}) \times \Omega$. Moreover $u \in C([0, T_{max}); H_{\Gamma_0}^1(\Omega))$,*

$$u_t \in L^m((0, T) \times \Gamma_1) \cap L^2((0, T) \times \Omega) \quad \text{for any } T \in (0, T_{max}),$$

and the following alternative holds:

- (i) either $T_{max} = \infty$;
- (ii) or $T_{max} < \infty$ and

$$(8) \quad \lim_{t \rightarrow T_{max}^-} \|u(t)\|_{H_{\Gamma_0}^1} = +\infty.$$

Finally u depends continuously on the initial datum u_0 , that is given any $T \in (0, T_{max})$ and any sequence $(u_{0n})_n$ in $H_{\Gamma_0}^1(\Omega)$ such that $u_{0n} \rightarrow u_0$ in $H_{\Gamma_0}^1(\Omega)$, the corresponding weak solution u^n is defined in $[0, T] \times \Omega$ and $u^n \rightarrow u$ in $C([0, T]; H_{\Gamma_0}^1(\Omega))$.

¹see Definition 2 below for the precise meaning of weak solutions, which are essentially distributional solutions enjoying a suitable regularity

The second aim of the paper is to study the alternative (i)–(ii) in previous Theorem by giving global existence versus blow-up results. When $p = 2$ it is straightforward to prove that u is global (see Theorem 11 in Appendix B), so we focus on the more interesting case $p > 2$. Although we are not able to give a complete answer, as usual for nonlinear problems, we give two partial answers when

$$(9) \quad \sigma(\Gamma_0) > 0,$$

so a Poincaré-type inequality holds (see [63]) and consequently $\|\nabla u\|_2$ is an equivalent norm in $H_{\Gamma_0}^1(\Omega)$. This assumption allows us to use potential-well arguments. In order to state our next results we need to recall the stable and unstable sets introduced in [58]. When $p > 2$ and (3) holds we introduce the functionals

$$(10) \quad J(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p} \|u\|_p^p, \quad K(u) = \|\nabla u\|_2^2 - \|u\|_p^p$$

defined for $u \in H_{\Gamma_0}^1(\Omega)$, and the number

$$(11) \quad d = \inf_{u \in H_{\Gamma_0}^1(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\lambda u).$$

When $p > 2$ and (3), (9) hold true it is easy to see that $d > 0$. See Lemma 2 below, where two different characterizations of d are given. We define the stable and unstable sets as

$$(12) \quad W_s = \{u_0 \in H_{\Gamma_0}^1(\Omega) : K(u_0) \geq 0 \text{ and } J(u_0) < d\}$$

$$(13) \quad W_u = \{u_0 \in H_{\Gamma_0}^1(\Omega) : K(u_0) \leq 0 \text{ and } J(u_0) < d\}.$$

As an application of Theorem 2 and of a potential-well estimate we give the following global existence result.

Theorem 3. (Global existence) *Under the assumptions of Theorem 1 and the further assumptions (9) and $p > 2$, if $u_0 \in W_s$ then $T_{max} = \infty$ and $u(t) \in W_s$ for all $t \geq 0$.*

While Theorem 3 can be seen as a simple application of Theorem 2, to recognize that solutions of problem (2) starting in the unstable set blow-up is a more difficult task. When $m = 2$ this result can be proved by a concavity argument (see [56]), which cannot be applied when $m \neq 2$, making this case more interesting. By combining the main technique of [34] with an estimate used in [58] for wave equation we are able to prove the following result.

Theorem 4. (Blow-up) *Under the assumptions of Theorem 1 and the further assumptions (9), $p > 2$ and*

$$(14) \quad m < m_0(p) := \frac{2(n+1)p - 4(n-1)}{n(p-2) + 4},$$

if $u_0 \in W_u$ then $T_{max} < \infty$, $u(t) \in W_u$ for all $t \in [0, T_{max})$, and $\lim_{t \rightarrow T_{max}^-} \|u(t)\|_p = +\infty$.

Remark 2. Even if is not evident from (12) and (13), $W_s \cap W_u = \emptyset$ (see Lemma 2 below), so Theorems 3 and 4 are consistent.

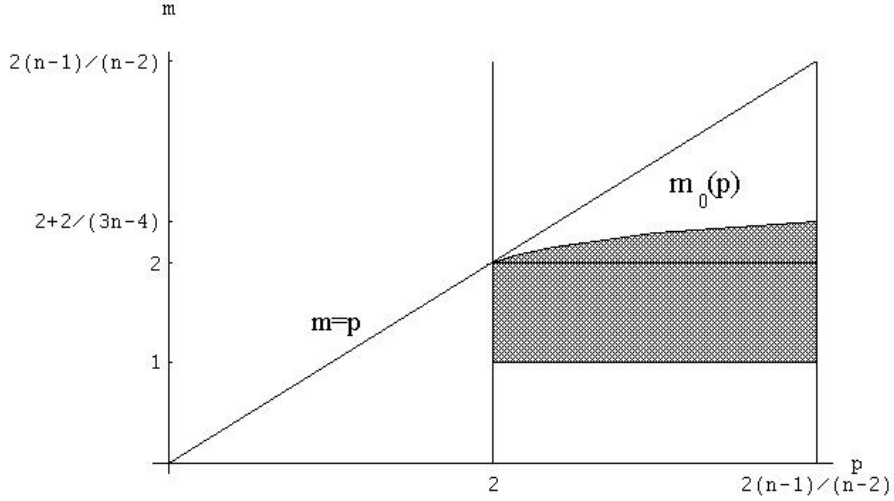


FIGURE 1. The shaded region is the set of the (p, m) couples for which the assumptions of Theorem 4 hold, when $n \geq 3$. The picture is made in the case $n = 3$.

Remark 3. Clearly assumption (14) yields $m < p$ since it is trivial to prove that $m_0(p) \leq p$ for $p \geq 2$. It strongly reduces the applicability of Theorem 4, as shown by Figure 1 which illustrates the set of the couples (p, m) satisfying (3) and (14). As $m_0(p) > 2$ for $p > 2$, the result is rather sharp in the sublinear case $1 < m \leq 2$, while (3) and (14) force that $m < 4$ when $n = 1$, $m < 3$ when $n = 2$ and $m < 2 + \frac{2}{3n-4}$ when $n \geq 3$. This assumption, which looks to be a technical one, comes directly from [58], where it was introduced, and is due to the difficulty in comparing the effect of high order polynomial dissipation, which is related to the L^m norm on Γ_1 , with the effect of the source, related to the L^p norm on Ω . After nine years from its use, the authors are not aware of any improvement.

As a preliminary step in the proof of Theorem 1 we give a well-posedness result for the problem

$$(15) \quad \begin{cases} u_t - \Delta u = g(t, x) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -|u_t|^{m-2} u_t & \text{on } [0, T) \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

where $m > 1$, $T > 0$ is arbitrary and g is a given forcing term acting on Ω . Although problem (15) can be studied using the analysis of [16], it is not trivial in that way to get the following result.

Theorem 5. (Well-posedness for an auxiliary problem) *Suppose that $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. Then there is a unique weak² solution u of (15)*

²see Definition 1 below for the precise meaning of weak solution

in $[0, T] \times \Omega$. Moreover

$$(16) \quad u \in C([0, T]; H_{\Gamma_0}^1(\Omega)),$$

$$(17) \quad u_t \in L^2((0, T) \times \Omega) \cap L^m((0, T) \times \Gamma_1)$$

and the energy identity

$$(18) \quad \frac{1}{2} \|\nabla u\|_2^2 \Big|_s^t + \int_s^t \|u_t\|_2^2 + \|u_t\|_{m, \Gamma_1}^m = \int_s^t \int_{\Omega} g u_t$$

holds for $0 \leq s \leq t \leq T$. Finally, given any couple of initial data $u_{01}, u_{02} \in H_{\Gamma_0}^1(\Omega)$ and any couple of forcing terms $g_1, g_2 \in L^2((0, T) \times \Omega)$, respectively denoting by u^1 and u^2 the solutions of (15) corresponding to u_{01}, g_1 and to u_{02}, g_2 , the following estimate holds

$$(19) \quad \|u^1 - u^2\|_{C([0, T]; H_{\Gamma_0}^1(\Omega))}^2 \leq 2(1+T) \left(\|u_{01} - u_{02}\|_{H_{\Gamma_0}^1}^2 + \|g_1 - g_2\|_{L^2((0, T) \times \Omega)}^2 \right).$$

Remark 4. A short comparison with the results which can be obtained by directly applying the abstract results in [16] is in order. Assumptions (A1–2) in [16, Theorem 1] force to restrict to the case $m = 2$, while the assumption $D(B) \subset V$ in [16, Theorems 2–3] implies $m \leq 2(n-1)/(n-2)$ when $n \geq 3$. Next one can apply [16, Theorem 4] only when g is more regular in time. Finally, [16, Theorem 5] can be applied only when $m = 2$.

In order to explain the main difficulties arising in the proofs of our main results we now make some comparison with the arguments used by the second author in [59]. Theorem 5 is essentially proved as [59, Theorem 1.5], even if the necessary adaptations require some care. Theorem 1 is proved by a contraction argument instead that a compactness one. Theorem 2 has no counterpart in [59]. Finally the proof of Theorem 4 requires an untrivial mixing of the technique of [34] with the estimate used in [58], so the authors consider it as the main contribution in the present paper.

The paper is organized as follows. Section 2 deals with some notation and preliminary material, including the proof of Theorem 5, Section 3 is devoted to local well-posedness theory for problem (2) while in Section 4 we study global existence and blow-up for it. Finally the results presented in this introduction are generalized in Section 5 to problem (1), under suitable assumptions on the nonlinearities f and Q . For the sake of simplicity we first present the proofs for the model problem (2) and then we give in (5) the generalizations needed to handle with (1). This section is naturally addressed to a more specialized audience and consequently an higher lever of mathematical expertise of the reader is supposed. In particular most proofs are only sketched.

2. NOTATION AND PRELIMINARIES

We introduce the notations

$C_c^\infty(\mathcal{O})$	space of compactly supported real-valued C^∞ functions on any open set $\mathcal{O} \subset \mathbb{R}^n$,
$C^\infty((a, b); X)$	space of C^∞ X -valued functions in (a, b) , X Banach space,
$C([a, b]; X)$	space of norm continuous X -valued functions in $[a, b]$,
$C_w([a, b]; X)$	space of weakly continuous X -valued functions in $[a, b]$,
q'	Hölder conjugate of $q \geq 1$, i.e. $1/q + 1/q' = 1$,
X'	the dual space of X ,
(\cdot, \cdot)	scalar product in $L^2(\Omega)$.

Moreover we call *the trace theorem* the existence of the continuous trace mapping $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^2(\partial\Omega)$. Moreover the trace of u on Ω will be denoted by $u|_{\partial\Omega}$. We also call *the Sobolev Embedding Theorem* the existence of the continuous embedding $H_{\Gamma_0}^1(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \leq p < 2^*$.

We start by setting the Banach space

$$(20) \quad X = \{u \in H_{\Gamma_0}^1(\Omega) : u|_{\Gamma_1} \in L^m(\Gamma_1)\}$$

endowed with the norm $\|u\|_X = \|u\|_{H_{\Gamma_0}^1} + \|u|_{\Gamma_1}\|_{m, \Gamma_1}$. For elements $u \in X$ we shall use the simpler notation $\|u\|_{m, \Gamma_1}$ to mean $\|u|_{\Gamma_1}\|_{m, \Gamma_1}$. We now give the precise meaning of weak solution of (15).

Definition 1. Let $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. We say that u is a weak solution of (15) in $[0, T] \times \Omega$ if

- (a) $u \in L^\infty(0, T; H_{\Gamma_0}^1(\Omega))$, $u_t \in L^2((0, T) \times \Omega)$;
 - (b) the spatial trace of u on $(0, T) \times \partial\Omega$ (which exists by the trace theorem) has a distributional time derivative on $(0, T) \times \partial\Omega$, belonging to $L^m((0, T) \times \partial\Omega)$;
 - (c) for all $\phi \in X$ and for almost all $t \in [0, T]$ the distribution identity
- $$(21) \quad \int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi + \int_{\Gamma_1} |u_t(t)|^{m-2} u_t(t) \phi = \int_{\Omega} g(t) \phi$$
- holds true;
- (d) $u(0) = u_0$.

Note that, in (d), $u(0)$ makes sense since, by (a),

$$u \in H^1(0, T; L^2(\Omega)) \hookrightarrow C([0, T]; L^2(\Omega)).$$

In order to prove Theorem 5 we need the following Lemma, which extends [53, Theorems 3.1 and 3.2] to the present situation. Its proof consists in a rather technical application of the arguments in [53] which is given in Appendix C for the reader convenience.

Lemma 1. Let $0 < T < \infty$, $m > 1$,

$$(22) \quad u_0 \in H_{\Gamma_0}^1(\Omega), \quad g \in L^2((0, T) \times \Omega), \quad \zeta \in L^{m'}((0, T) \times \Gamma_1)$$

and suppose that u is a weak solution of

$$(23) \quad \begin{cases} u_t - \Delta u = g(t, x) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = \zeta & \text{on } [0, T] \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

i.e. a function

$$(24) \quad u \in L^\infty(0, T; H_{\Gamma_0}^1(\Omega))$$

such that

$$(25) \quad u_t \in L^2((0, T) \times \Omega),$$

the spatial trace of u on $(0, T) \times \partial\Omega$ (which exists by the trace theorem) has a distributional time derivate on $(0, T) \times \partial\Omega$ belonging to $L^m((0, T) \times \partial\Omega)$, and, for all $\phi \in X$ and almost all $t \in [0, T]$ the function u satisfies

$$(26) \quad \int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi - \int_{\Gamma_1} \zeta(t) \phi = \int_{\Omega} g(t) \phi.$$

Then

$$(27) \quad u \in C([0, T]; H_{\Gamma_0}^1(\Omega))$$

and the energy identity

$$(28) \quad \frac{1}{2} \|\nabla u\|_2^2 \Big|_s^t + \int_s^t \|u_t\|_2^2 - \int_s^t \int_{\Gamma_1} \zeta u_t = \int_s^t \int_{\Omega} g u_t$$

holds for $0 \leq s \leq t \leq T$.

Proof of Theorem 5. To prove the existence of a weak solution of (15) we apply the Faedo–Galerkin procedure. Let $(w_k)_k$ be a sequence of linearly independent vectors in the space X , which was defined in (20), whose finite linear combinations are dense in it. By using the Graham–Schmidt orthonormalization process, we can take $(w_k)_k$ to be orthonormal in $L^2(\Omega)$. Since (see [57, Lemma A1, Appendix A]) X is dense in $H_{\Gamma_0}^1(\Omega)$ for all $k \in \mathbb{N}$ there are real numbers y_{0k}^j , $j = 1, \dots, k$, such that

$$(29) \quad u_{0k} = \sum_{j=1}^k y_{0k}^j w_j \rightarrow u_0 \text{ in } H_{\Gamma_0}^1(\Omega).$$

For any fixed $k \in \mathbb{N}$ we look for approximate solutions of (15), that is for solutions

$u^k(t) = \sum_{j=1}^k y_k^j(t) w_j$, of the finite-dimensional problem

$$(30) \quad \begin{cases} (u_t^k, w_j) + (\nabla u_t^k, \nabla w_j) + \int_{\Gamma_1} |u_t^k|^{m-2} u_t^k w_j = \int_{\Omega} g w_j, & j = 1, \dots, k, \\ u^k(0) = u_{0k}. \end{cases}$$

In order to recognize that (30) has a local solution, we set

$$(31) \quad y_{0k} = (y_{0k}^1, \dots, y_{0k}^k)^T, \quad y_k = (y_k^1, \dots, y_k^k)^T,$$

$$(32) \quad A_k = ((\nabla w_i, \nabla w_j))_{i,j=1,\dots,k}, \quad B_k(x) = (w_1(x), \dots, w_k(x))^T,$$

$$(33) \quad G_k(y) = y + \int_{\Gamma_1} |B_k(x) \cdot y|^{m-2} B_k(x) \cdot y B_k(x) dx, \quad y \in \mathbb{R}^k,$$

and $H_k(t) = \int_{\Omega} g(t, x) B_k(x) dx$, so problem (30) can be rewritten as

$$(34) \quad \begin{cases} G_k(y_k'(t)) + A_k y_k(t) = H_k(t), \\ y_k(0) = y_{0k}. \end{cases}$$

Then, using the arguments in [59, Proof of Theorem 1.5] we get that G_k is an homeomorphism from \mathbb{R}^k into itself, with inverse G_k^{-1} , and that (34) has a solution $y_k \in W^{1,1}(0, t_k)$ for some $t_k \in (0, T]$, and consequently (30) has a solution $u^k \in$

$W^{1,1}(0, t_k; X)$. Moreover, since $G_k(y)y \geq |y|^2$ for all $y \in \mathbb{R}^k$, by the Schwartz inequality it follows that $|y| \leq |G_k(y)|$. Then $|G_k^{-1}(y)| \leq |y|$ for all $y \in \mathbb{R}^k$, so that

$$(35) \quad |G_k^{-1}(H_k(t) - A_k y_k(t))| \leq |H_k(t)| + \|A_k\| |y_k|.$$

Multiplying (30) by $(y_k^j)'$ and summing for $j = 1, \dots, k$, we obtain the energy identity (here and in the sequel, explicit dependence on t will be omitted, when clear)

$$(36) \quad \frac{d}{dt} \left(\frac{1}{2} \|\nabla u^k\|_2^2 \right) + \|u_t^k\|_2^2 + \|u_t^k\|_{m, \Gamma_1}^m = \int_{\Omega} g u_t^k.$$

Integrating over $(0, t)$, $0 < t < t_k$, and using Young inequality, we get

$$\frac{1}{2} \|\nabla u^k\|_2^2 + \int_0^t \left(\|u_t^k\|_2^2 + \|u_t^k\|_{m, \Gamma_1}^m \right) \leq \frac{1}{2} \|\nabla u_0\|_2^2 + \frac{1}{2} \|g\|_{L^2((0, T) \times \Omega)}^2 + \frac{1}{2} \int_0^t \|u_t^k\|_2^2.$$

Then, using (29), there exists $C = C(\|\nabla u_0\|_2, \|g\|_{L^2((0, T) \times \Omega)}) > 0$ such that

$$(37) \quad \begin{aligned} \|\nabla u^k\|_{L^\infty(0, t_k; L^2(\Omega))} &\leq C, \\ \|u_t^k\|_{L^2((0, t_k) \times \Omega)} &\leq C, \\ \|u_t^k\|_{L^m((0, t_k) \times \Gamma_1)} &\leq C, \\ \| |u_t^k|^{m-2} u_t^k \|_{L^{m'}((0, t_k) \times \Gamma_1)} &\leq C, \end{aligned}$$

for $k \in \mathbb{N}$. By (29), (37) and Hölder inequality in time it follows that

$$(38) \quad \|u^k\|_2 \leq \|u_0\|_2 + \int_0^t \|u_t^k\|_2 \leq \|u_0\|_2 + T^{1/2} \left(\int_0^t \|u_t^k\|_2^2 \right)^{1/2} \leq C'$$

for some $C' = C'(\|u_0\|_{H_{\Gamma_0}^1}, \|g\|_{L^2((0, T) \times \Omega)}, T) > 0$. Since $(w_k)_k$ is orthonormal in $L^2(\Omega)$, we have $|y_k(t)| = \|u^k(t)\|_2$, so (38) yields that $|y_k(t)| \leq C'$. Then, by (35)

$$|G_k^{-1}(H_k(t) - A_k y_k(t))| \leq |H_k(t)| + C' \|A_k\| \in L^1(0, T).$$

We can then apply [15, Theorem 1.3, Chapter 2] to conclude that $t_k = T$ for $k = 1, \dots, n$. Next, by (37) and (38), it follows that, up to a subsequence,

$$(39) \quad \begin{aligned} u^k &\rightarrow u \quad \text{weakly* in } L^\infty(0, T; H_{\Gamma_0}^1(\Omega)), \\ u_t^k &\rightarrow u_t \quad \text{weakly in } L^2((0, T) \times \Omega), \\ u_t^k &\rightarrow \varphi \quad \text{weakly in } L^m((0, T) \times \Gamma_1), \\ |u_t^k|^{m-2} u_t^k &\rightarrow \chi \quad \text{weakly in } L^{m'}((0, T) \times \Gamma_1). \end{aligned}$$

A consequence of the convergences (39) and of Aubin–Lions compactness Lemma (see [10, 3, 52]) is that $u^k \rightarrow u$ strongly in $C([0, T]; L^2(\Omega))$, so that $u(0) = u_0$. It follows in a standard way (see, for example, [57, p. 272]) that φ is the distribution time derivate of u on $(0, T) \times \partial\Omega$, i.e. $\varphi = u_t$.

Next, multiplying (30) by $\phi \in C_c^\infty(0, T)$, integrating on $(0, T)$, passing to the limit as $k \rightarrow \infty$ (using (39)) and finally using the density of the finite linear combinations of $(w_k)_k$ in X , we obtain $\int_0^T [(u_t, w) + (\nabla u_t, \nabla w) + \int_{\Gamma_1} \chi w - \int_{\Omega} g w] \phi = 0$ for all $w \in X$, $\phi \in C_c^\infty(0, T)$. Consequently $(u_t, w) + (\nabla u, \nabla w) + \int_{\Gamma_1} \chi w = \int_{\Omega} g w$ almost

everywhere in $(0, T)$. Then to prove that u is a weak solution of (15) we have only to show that

$$(40) \quad \chi = |u_t|^{m-2} u_t \text{ a.e. on } (0, T) \times \Gamma_1.$$

By Lemma 1 we obtain (27) and the energy identity

$$(41) \quad \frac{1}{2} \|\nabla u\|_2^2 \Big|_0^T + \int_0^T \|u_t\|_2^2 + \int_0^T \int_{\Gamma_1} \chi u_t = \int_0^T \int_{\Omega} g u_t$$

The classical monotonicity method (see [41] or [59, p. 186]) then allows us to prove (40).

Finally, to prove the estimate (19), which also yields the uniqueness of the solution, we recognize that $v = u_1 - u_2$ is a weak solution of problem (23) with $g = g_1 - g_2$, $\xi = -|u_t^1|^{m-2} u_t^1 + |u_t^2|^{m-2} u_t^2$ and $u_0 = u_{01} - u_{02}$. Then, by Lemma 1, using the monotonicity of the map $x \rightarrow |x|^{m-2} x$ we get the estimate

$$\frac{1}{2} \|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 \leq \int_0^t g v_t + \frac{1}{2} \|\nabla u_0\|_2^2 \quad \text{for all } t \in [0, T].$$

By Young inequality

$$\|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 \leq \|g\|_{L^2((0,T) \times \Omega)}^2 + \|\nabla u_0\|_2^2 \quad \text{for all } t \in [0, T].$$

Moreover $\|v(t)\|_2^2 \leq \left(\|u_0\|_2 + \int_0^t \|v_t\|_2 \right)^2 \leq 2\|u_0\|_2^2 + 2T \int_0^t \|v_t\|_2^2$. By combining the last two estimates we get (19) and conclude the proof. \square

3. PROOFS OF THEOREMS 1 AND 2.

This section is devoted to prove our main well-posedness Theorems 1 and 2. We first precise the meaning of weak solution for problem (2).

Definition 2. Let $u_0 \in H_{\Gamma_0}^1(\Omega)$. When assumption (3) holds we say that u is a weak solution of problem (2) in $[0, T] \times \Omega$ if (a–d) of Definition 1 hold, with the distribution identity (21) being replaced by

$$(42) \quad \int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi + \int_{\Gamma_1} |u_t(t)|^{m-2} u_t(t) \phi = \int_{\Omega} |u(t)|^{p-2} u(t) \phi.$$

Moreover we say that u is a weak solution of problem (2) in $[0, T] \times \Omega$ if u is a weak solution of (2) in $[0, T'] \times \Omega$ for all $T' \in (0, T)$.

Remark 5. Since $p \leq 1 + 2^*/2$ and $\varphi \in H_{\Gamma_0}^1(\Omega)$ the integral in the right-hand side of (42) makes sense due to the Sobolev Embedding Theorem.

Proof of Theorem 1. We set, for any $0 < T < \infty$, the Banach space $Y_T = C([0, T]; H_{\Gamma_0}^1(\Omega))$ endowed with the usual norm $\|u\|_{Y_T} = \|u\|_{L^\infty(0,T; H_{\Gamma_0}^1(\Omega))}$, and the closed convex set $X_T = \{u \in Y_T : u(0) = u_0\}$. Let $u \in X_T$. By (3) we have $2(p-1) \leq 2^*$ and then, by the Sobolev Embedding Theorem,

$$(43) \quad \|u(t)\|_{2(p-1)} \leq K_0 \|u(t)\|_{H_{\Gamma_0}^1}, \quad \forall t \in [0, T],$$

for some $K_0 = K_0(\Omega) > 0$ (in the sequel of the proof K_i , $i \in \mathbb{N}$, will denote suitable positive constants depending on p , n and Ω). Hence $|u|^{p-2}u \in L^\infty((0, T); L^2(\Omega))$. Then by Theorem 5 there is a unique weak solution v of the problem

$$(44) \quad \begin{cases} v_t - \Delta v = |u|^{p-2}u & \text{in } (0, T) \times \Omega, \\ v = 0 & \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial v}{\partial \nu} = -|v_t|^{m-2}v_t & \text{on } [0, T] \times \Gamma_1, \\ v(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Moreover $v \in C([0, T]; H_{\Gamma_0}^1(\Omega))$, $v_t \in L^m((0, T) \times \Gamma_1) \cap L^2((0, T) \times \Omega)$ and the energy identity

$$(45) \quad \frac{1}{2} \|\nabla v\|_2^2 \Big|_0^t + \int_0^t \left(\|v_t\|_{m, \Gamma_1}^m + \|v_t\|_2^2 \right) = \int_0^t \int_\Omega |u|^{p-2} u v_t$$

holds for all $t \in [0, T]$. We define $\Phi : X_T \rightarrow X_T$ by $\Phi(u) = v$, where v denotes the solution of (44) that corresponds to u . We are going to prove that we can apply the Banach Contraction Theorem to $\Phi : B_R \rightarrow B_R$ where $B_R = \{u \in X_T : \|u\|_{Y_T} \leq R\}$, provided that R is sufficiently large and T is sufficiently small. Note that B_R is non-empty for

$$(46) \quad R \geq R_0 := \|u_0\|_{H_{\Gamma_0}^1}.$$

We first claim that Φ maps B_R into itself for R sufficiently large and T small enough. Let $u \in B_R$. By (45) and (43) we get, for $t \in [0, T]$,

$$\frac{1}{2} \|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 \leq \frac{1}{2} \|\nabla u_0\|_2^2 + K_0^{2(p-1)} \int_0^t \|u\|_{H_{\Gamma_0}^1}^{p-1} \|v_t\|_2.$$

Now using Young inequality it follows that, for all $t \in [0, T]$,

$$\begin{aligned} \frac{1}{2} \|\nabla v(t)\|_2^2 + \int_0^t \|v_t\|_2^2 &\leq \frac{1}{2} \|\nabla u_0\|_2^2 + K_1 R^{p-1} \int_0^t \|v_t\|_2 \\ &\leq \frac{1}{2} \|\nabla u_0\|_2^2 + \frac{1}{2} K_1^2 R^{2(p-1)} T + \frac{1}{2} \int_0^t \|v_t\|_2^2. \end{aligned}$$

Hence

$$(47) \quad \frac{1}{2} \|\nabla v(t)\|_2^2 + \frac{1}{2} \int_0^t \|v_t\|_2^2 \leq \frac{1}{2} \|\nabla u_0\|_2^2 + K_2 R^{2(p-1)} T.$$

Consequently, by (46),

$$(48) \quad \|\nabla v\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq R_0^2 + 2K_2 R^{2(p-1)} T$$

and

$$(49) \quad \int_0^T \|v_t\|_2^2 \leq R_0^2 + 2K_2 R^{2(p-1)} T.$$

Using Hölder inequality we have $\|v(t)\|_2 = \left\| u_0 + \int_0^t v_t(s) ds \right\|_2 \leq \|u_0\|_2 + T^{\frac{1}{2}} \left(\int_0^t \|v_t\|_2^2 \right)^{\frac{1}{2}}$ and so, by (49),

$$(50) \quad \|v(t)\|_2^2 \leq 2\|u_0\|_2^2 + 2T \left(\int_0^t \|v_t\|_2^2 \right) \leq 2(1+T)R_0^2 + 4K_2 R^{2(p-1)} T^2.$$

Now restricting to $T \leq 1$ we have $T^2 \leq T$ and so combining (48) and (50) we get

$$(51) \quad \|v\|_{Y_T}^2 \leq (3 + 2T)R_0^2 + 6K_2R^{2(p-1)}T \leq 5R_0^2 + 6K_2R^{2(p-1)}T.$$

By (51) in order to prove that $v \in B_R$, it is enough to show that $5R_0^2 \leq \frac{1}{2}R^2$ and $6K_2R^{2(p-1)}T \leq \frac{1}{2}R^2$. Hence our claim holds for

$$(52) \quad R = 4R_0 \quad \text{and} \quad T \leq \min \left\{ 1, K_3R_0^{2(2-p)} \right\}.$$

In the sequel we shall assume that (52) holds.

We now claim that, for T small enough, the map Φ is a contraction. Let $u, \bar{u} \in B_R$, and denote $v = \Phi(u)$, $\bar{v} = \Phi(\bar{u})$, $w = v - \bar{v}$. Clearly, w is a weak solution (in the sense of Lemma 1) of the problem

$$(53) \quad \begin{cases} w_t - \Delta w = |u|^{p-2}u - |\bar{u}|^{p-2}\bar{u} & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial w}{\partial \nu} = -|v_t|^{m-2}v_t + |\bar{v}_t|^{m-2}\bar{v}_t & \text{on } [0, T) \times \Gamma_1, \\ w(0, x) = 0 & \text{in } \Omega. \end{cases}$$

Since $v_t, \bar{v}_t \in L^m((0, T) \times \Gamma_1)$, we also know that $|v_t|^{m-2}v_t$ and $|\bar{v}_t|^{m-2}\bar{v}_t$ belong to $L^{m'}((0, T) \times \Gamma_1)$. Moreover, by (3), the functions $|u|^{p-2}u$ and $|\bar{u}|^{p-2}\bar{u}$ belong to $L^2((0, T) \times \Omega)$. Then we can apply Lemma 1 so that, for $t \in [0, T]$,

$$(54) \quad \begin{aligned} \frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 + \int_0^t \int_{\Gamma_1} \left[|v_t|^{m-2}v_t - |\bar{v}_t|^{m-2}\bar{v}_t \right] w_t \\ = \int_0^t \int_{\Omega} \left[|u|^{p-2}u - |\bar{u}|^{p-2}\bar{u} \right] w_t. \end{aligned}$$

Using the monotonicity of the map $x \rightarrow |x|^{m-2}x$ and the elementary inequality

$$(55) \quad \left| |A|^{p-2}A - |B|^{p-2}B \right| \leq K_4 |A - B| \left(|A|^{p-2} + |B|^{p-2} \right),$$

for $A, B \in \mathbb{R}$, $p \geq 2$, we get

$$(56) \quad \frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 \leq K_4 \int_0^t \int_{\Omega} \left(|u|^{p-2} + |\bar{u}|^{p-2} \right) |u - \bar{u}| |w_t|.$$

We now set $r = 2^*$ if $n \in \mathbb{N}$, $n \neq 2$, while $r = 2p$ when $n = 2$, so that $2 \leq p \leq 1 + r/2 \leq 1 + 2^*/2$ and $r > 2$. We also fix $s > 2$ such that $\frac{1}{s} + \frac{1}{r} + \frac{1}{2} = 1$, that is $s = \frac{2r}{r-2}$. By applying triple Hölder inequality and the elementary inequality

$$(57) \quad (A + B)^\tau \leq \max \{1, 2^{\tau-1}\} (A^\tau + B^\tau) \quad \text{for } A, B \geq 0, \quad \tau \geq 0,$$

from (56) we get

$$\frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 \leq K_5 \int_0^t \left(\int_{\Omega} \left(|u|^{s(p-2)} + |\bar{u}|^{s(p-2)} \right) \right)^{\frac{1}{s}} \|u - \bar{u}\|_r \|w_t\|_2.$$

But $s(p-2) \leq r$ since $p \leq 1 + \frac{r}{2}$, so by the Sobolev Embedding Theorem and weighted Young inequality we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
 \frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 &\leq K_6 \int_0^t \left[\|u\|_r^{s(p-2)} + \|\bar{u}\|_r^{s(p-2)} \right]^{\frac{1}{s}} \|u - \bar{u}\|_r \|w_t\|_2 \\
 &\leq K_7 R^{p-2} \int_0^t \|u - \bar{u}\|_r \|w_t\|_2 \\
 &\leq K_8 R^{p-2} \int_0^t \|u - \bar{u}\|_{H_{\Gamma_0}^1} \|w_t\|_2 \\
 &\leq K_8 R^{p-2} \left[\frac{1}{2\varepsilon} \int_0^t \|u - \bar{u}\|_{H_{\Gamma_0}^1}^2 + \frac{\varepsilon}{2} \int_0^t \|w_t\|_2^2 \right]
 \end{aligned} \tag{58}$$

and consequently

$$\frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 \leq \frac{K_9 R^{p-2}}{\varepsilon} T \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}^2 + K_9 R^{p-2} \varepsilon \int_0^t \|w_t\|_2^2.$$

Now we choose $\varepsilon = 1/(2K_9 R^{p-2})$ so previous estimate reads as

$$\frac{1}{2} \|\nabla w(t)\|_2^2 + \frac{1}{2} \int_0^t \|w_t\|_2^2 \leq 2K_9^2 R^{2(p-2)} T \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}^2, \tag{59}$$

and consequently

$$\|\nabla w(t)\|_2 \leq K_{10} R^{p-2} \sqrt{T} \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))} \quad \text{for all } t \in [0, T] \tag{60}$$

and

$$\int_0^T \|w_t\|_2^2 \leq K_{11} R^{2(p-2)} T \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}^2. \tag{61}$$

Then, since $w(0) = 0$, by Hölder inequality and (61)

$$\|w(t)\|_2 \leq \int_0^t \|w_t\|_2 \leq T^{\frac{1}{2}} \left(\int_0^t \|w_t\|_2^2 \right)^{\frac{1}{2}} \leq T K_{12} R^{p-2} \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}. \tag{62}$$

By combining (60) and (62) we consequently get (as $T \leq 1$)

$$\|w(t)\|_{H_{\Gamma_0}^1}^2 \leq K_{13}^2 R^{2(p-2)} T \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}^2. \tag{63}$$

Then Φ is a contraction provided $K_{13} R^{p-2} \sqrt{T} < 1$, that is, by (52), provided

$$T < K_{13}^{-2} (4R_0)^{2(2-p)}. \tag{64}$$

We can finally choose $T^* = \min \left\{ 1, K_3 R_0^{2(2-p)}, \frac{1}{2} K_{13}^{-2} (4R_0)^{2(2-p)} \right\}$ which is decreasing in R_0 . So, by applying Banach Contraction Theorem with $T = T^*$, there is a weak solution of (2) on $[0, T^*] \times \Omega$ satisfying (4)–(6). Moreover (7) follows by (52).

In order to prove that the solution is unique we use a standard procedure of ODEs, using previous claims, which is briefly outlined as follows. Let u, \tilde{u} be two weak solutions of (2) on $[0, T^*] \times \Omega$. By Lemma 1 we have $u, \tilde{u} \in C([0, T^*]; H_{\Gamma_0}^1(\Omega))$. Suppose by contradiction that $u \neq \tilde{u}$. Then

$$(65) \quad T' = \sup \{ \tau > 0 : u = \tilde{u} \text{ on } [0, \tau] \} < T^* \quad \text{and } u(T') = \tilde{u}(T') \text{ by continuity.}$$

Setting $u_1(t) = u(t+T')$, $\tilde{u}_1(t) = \tilde{u}(t+T')$ we have $u_1, \tilde{u}_1 \in C([0, T^* - T']; H_{\Gamma_0}^1(\Omega))$ and $\tilde{u}_0 := u_1(0) = \tilde{u}_1(0)$. Then u_1, \tilde{u}_1 are weak solutions of (2) with initial datum \tilde{u}_0 . By continuity there is $0 < T'' \leq T^* - T'$ such that

$$\max\{\|u_1\|_{C([0, T'']; H_{\Gamma_0}^1(\Omega))}, \|\tilde{u}_1\|_{C([0, T'']; H_{\Gamma_0}^1(\Omega))}\} \leq 4 \|\tilde{u}_0\|_{H_{\Gamma_0}^1}.$$

Hence u_1 and \tilde{u}_1 are fixed points for Φ in $B_{4\|\tilde{u}_0\|_{H_{\Gamma_0}^1}}$ when $T = T''$, so by previous claim $u_1 = \tilde{u}_1$ on $[0, T]$, contradicting (65). \square

Proof of Theorem 2. The existence of the unique maximal solution u of (2) follows by Theorem 1 in a standard way: first one sets \mathcal{U} to be the set of all tweak solutions of (2), then one proves that any two elements of \mathcal{U} must coincide on the intersection of their domains, arguing as at the end of previous proof, finally one defines $u(t)$ to coincide with any of these solution for t in the union of the domains.

Next, in order to prove that the alternative (i)–(ii) holds, let us suppose, by contradiction, that

$$(66) \quad T_{\max} < \infty \quad \text{and} \quad \overline{\lim}_{t \rightarrow T_{\max}} \|u(t)\|_{H_{\Gamma_0}^1} < \infty.$$

Then there is a sequence $T_n \rightarrow T_{\max}^-$ such that $\|u(T_n)\|_{H_{\Gamma_0}^1}$ is bounded. Thus, by Theorem 1, the Cauchy problem (2) with initial time T_n and initial datum $u(T_n)$ as a unique weak solution in $[T_n, T_n + T']$, where $T' = T^*(\sup_{n \in \mathbb{N}} \|u(T_n)\|_{H_{\Gamma_0}^1}, m, p, \Omega, \Gamma_1)$ is independent on n . This leads to a contradiction, since, in this way, we can continue the solution to the right of T_{\max} .

Now, in order to prove that u depends continuously on the initial datum, we fix $T \in (0, T_{\max})$ and we denote $M = \|u\|_{C([0, T]; H_{\Gamma_0}^1(\Omega))}$. Since $u_{0n} \rightarrow u_0$ in $H_{\Gamma_0}^1(\Omega)$ there is $n_1 \in \mathbb{N}$ such that $\|u_{0n}\|_{H_{\Gamma_0}^1} \leq \|u_0\|_{H_{\Gamma_0}^1} + 1 \leq M + 1$. Then, by Theorem 1, problem (2) with initial datum u_{0n} has an unique solution u^n in $[0, T^*] \times \Omega$, with $T^* = T^*(M + 1, m, p, \Omega, \Gamma_1) \in (0, 1]$ and

$$(67) \quad \|u^n\|_{C([0, T^*]; H_{\Gamma_0}^1(\Omega))} \leq 4 \|u_{0n}\|_{H_{\Gamma_0}^1} \leq 4(M + 1)$$

for all $n \in \mathbb{N}$. Now we define $w^n = u^n - u$, which is a weak solution of the problem

$$\begin{cases} w_t^n - \Delta w^n = |u^n|^{p-2} u^n - |u|^{p-2} u & \text{in } (0, T^*) \times \Omega, \\ w^n = 0 & \text{on } [0, T^*) \times \Gamma_0, \\ \frac{\partial w^n}{\partial \nu} = -|u_t^n|^{m-2} u_t^n + |u_t|^{m-2} u_t & \text{on } [0, T^*) \times \Gamma_1, \\ w^n(0) = u_{0n} - u_0 & \text{in } \Omega \end{cases}$$

in the sense of Lemma 1. Consequently

$$(68) \quad \frac{1}{2} \|\nabla w^n\|_2^2 \Big|_0^t + \int_0^t \|w_t^n\|_2^2 + \int_0^t \int_{\Gamma_1} [|u_t^n|^{m-2} u_t^n - |u_t|^{m-2} u_t] w_t^n \\ = \int_0^t \int_{\Omega} [|u^n|^{p-2} u^n - |u|^{p-2} u] w_t^n.$$

Then, keeping the notation of the proof of Theorem 1 and using the arguments already used to prove (58) together with (67) we get the estimate

$$(69) \quad \frac{1}{2} \|\nabla w^n(t)\|_2^2 + \int_0^t \|w_t^n\|_2^2 \leq 4^{p-2}(M+1)^{p-2}K_8 \left[\frac{1}{2\varepsilon} \int_0^t \|w^n\|_{H_{\Gamma_0}^1}^2 + \frac{\varepsilon}{2} \|w_t^n\|_2^2 \right] + \frac{1}{2} \|\nabla(u_{0n} - u_0)\|_2^2$$

for any $\varepsilon > 0$. Consequently, for $\varepsilon > 0$ sufficiently small we have

$$(70) \quad \frac{1}{2} \|\nabla w^n(t)\|_2^2 + \frac{1}{2} \int_0^t \|w_t^n\|_2^2 \leq C_3 \int_0^t \|w^n\|_{H_{\Gamma_0}^1}^2 + \frac{1}{2} \|\nabla(u_{0n} - u_0)\|_2^2$$

where $C_3 = C_3(p, n, \Omega, u_0, T) > 0$. Moreover, since $T^* \leq 1$, by using Hölder inequality we get $\|w^n(t)\|_2 \leq \|u_{0n} - u_0\|_2 + \left(\int_0^t \|w_t^n\|_2^2 \right)^{1/2}$ and so by (70)

$$(71) \quad \|w^n(t)\|_2^2 \leq 2\|u_{0n} - u_0\|_2^2 + 2 \int_0^t \|w_t^n\|_2^2 \leq 2\|u_{0n} - u_0\|_2^2 + 4C_3 \int_0^t \|w^n\|_{H_{\Gamma_0}^1}^2.$$

Combining (70) and (71) we get

$$(72) \quad \|w^n(t)\|_{H_{\Gamma_0}^1}^2 \leq 2\|u_{0n} - u_0\|_{H_{\Gamma_0}^1}^2 + C_4 \int_0^t \|w^n\|_{H_{\Gamma_0}^1}^2$$

where $C_4 = C_4(p, n, \Omega, u_0, T) > 0$. By Gronwall inequality the estimate

$$(73) \quad \|w^n(t)\|_{H_{\Gamma_0}^1} \leq \sqrt{2} \|u_{0n} - u_0\|_{H_{\Gamma_0}^1} e^{\frac{C_4}{2}t}, \quad \text{for all } t \in [0, T^*],$$

follows. In particular we have

$$(74) \quad \|w^n(T^*)\|_{H_{\Gamma_0}^1} \leq \sqrt{2} \|u_{0n} - u_0\|_{H_{\Gamma_0}^1} e^{\frac{C_4}{2}T^*}.$$

Then, since $u_{0n} \rightarrow u_0$ as $n \rightarrow \infty$, for $n \geq n_2$, with n_2 sufficiently large, we have $\|u^n(T^*)\|_{H_{\Gamma_0}^1} \leq \|u(T^*)\|_{H_{\Gamma_0}^1} + 1 \leq M + 1$. Hence we get that u^n is defined in $[T^*, 2T^*]$. Moreover, by repeating previous argument for $t \in [T^*, 2T^*]$ and using (74), we get $\|w^n(t)\|_{H_{\Gamma_0}^1} \leq \sqrt{2} \|w^n(T^*)\|_{H_{\Gamma_0}^1} e^{\frac{C_4}{2}(t-T^*)} \leq 2\|u_{0n} - u_0\|_{H_{\Gamma_0}^1} e^{\frac{K_{15}}{2}t}$. After a finite number $k = \lceil \frac{T}{T^*} \rceil$ of iterations we get that for n large enough u^n is defined in $[0, T]$ and $\|u^n(t) - u(t)\|_{H_{\Gamma_0}^1} \leq 2^{\frac{k}{2}} \|u_{0n} - u_0\|_{H_{\Gamma_0}^1} e^{\frac{C_4}{2}t}$ for $t \in [0, T]$, concluding the proof. \square

4. PROOFS OF THEOREMS 3 AND 4.

When $\sigma(\Gamma_0) > 0$ a Poincarè type inequality holds (see [63, Corollary 4.5.3]) and we can take $\|\nabla u\|_2$ as an equivalent norm in $H_{\Gamma_0}^1(\Omega)$. Then using the Sobolev's Embedding Theorem, since $p \leq 1 + 2^*/2 \leq 2^*$, we have

$$(75) \quad B_1 := \sup_{u \in H_{\Gamma_0}^1(\Omega), u \neq 0} \frac{\|u\|_p}{\|\nabla u\|_2} < +\infty.$$

We denote, when $2 < p \leq 1 + 2^*/2$,

$$(76) \quad \lambda_1 = B_1^{-\frac{p}{p-2}}, \quad \widetilde{\lambda}_1 = B_1^{-\frac{2}{p-2}}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

$$(77) \quad W_1 = \{u_0 \in H_{\Gamma_0}^1(\Omega) : J(u_0) < E_1 \text{ and } \|\nabla u_0\|_2 < \lambda_1\},$$

$$(78) \quad \widetilde{W}_1 = \{u_0 \in H_{\Gamma_0}^1(\Omega) : J(u_0) < E_1 \text{ and } \|u_0\|_p < \widetilde{\lambda}_1\},$$

$$(79) \quad W_2 = \{u_0 \in H_{\Gamma_0}^1(\Omega) : J(u_0) < E_1 \text{ and } \|\nabla u_0\|_2 > \lambda_1\},$$

and

$$(80) \quad \widetilde{W}_2 = \{u_0 \in H_{\Gamma_0}^1(\Omega) : J(u_0) < E_1 \text{ and } \|u_0\|_p > \widetilde{\lambda}_1\}.$$

At first we give the following useful characterization of d , W_s and W_u .

Lemma 2. *Suppose $2 < p \leq 1 + 2^*/2$, $\sigma(\Gamma_0) > 0$ and let d , W_s and W_u be respectively defined by (11), (12) and (13). Then $E_1 = d$, $W_s = W_1 = \widetilde{W}_1$ and $W_u = W_2 = \widetilde{W}_2$.*

Proof. An easy calculation shows that for any $u \in H_{\Gamma_0}^1(\Omega) \setminus \{0\}$ we have $\max_{\lambda > 0} J(\lambda u) =$

$$J(\lambda(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\nabla u\|_2}{\|u\|_p}\right)^{2p/(p-2)} \text{ where } \lambda(u) = \frac{\|\nabla u\|_2^{2/(p-2)}}{\|u\|_p^{p/(p-2)}}. \text{ Hence, by (75),}$$

$d = E_1$. In order to show that $W_s = W_1 = \widetilde{W}_1$ we first prove that $W_s \subseteq W_1$. Let $u_0 \in W_s$ and suppose, by contradiction, that $\|\nabla u_0\| \geq \lambda_1$. Since $J(u_0) < d = E_1$ and $\|u_0\|_p^p \leq \|\nabla u_0\|_2^2$ it follows that

$$E_1 > J(u_0) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u_0\|_2^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

which contradicts (76). By (75), since $\widetilde{\lambda}_1 = B_1 \lambda_1$, one immediately gets that $W_1 \subseteq \widetilde{W}_1$. To prove that $\widetilde{W}_1 \subseteq W_s$, let $u_0 \in \widetilde{W}_1$. By (75), (78) and (76) we have $\|u_0\|_p^p < \widetilde{\lambda}_1^{p-2} \|u_0\|_p^2 = B_1^{-2} \|\nabla u_0\|_2^2$ and so $K(u_0) \geq 0$.

In order to show that $W_u = W_2 = \widetilde{W}_2$ we first prove that $W_2 \subseteq W_u$. Let $u_0 \in W_2$ and suppose, by contradiction, that $K(u_0) > 0$. So $\|u_0\|_p^p < \|\nabla u_0\|_2^2$ by (10). Moreover, $J(u_0) < d = E_1$ and $\|\nabla u_0\|_2 > \lambda_1$. Then it follows that

$$E_1 > \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u_0\|_2^2 > \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

which contradicts (76). By (75) one immediately gets that $\widetilde{W}_2 \subseteq W_2$. To prove that $W_u \subseteq \widetilde{W}_2$ and conclude the proof, we take $u_0 \in W_u$. We note that, by (75), we have $J(v) \geq h(\|v\|_p)$ for all $v \in H_{\Gamma_0}^1(\Omega)$, where h is defined by $h(\lambda) = \frac{1}{2} B_1^{-2} \lambda^2 - \frac{1}{p} \lambda^p$ for $\lambda \geq 0$. Moreover one easily verify that $h(\widetilde{\lambda}_1) = E_1$. The, since $J(u_0) < E_1$, we have $\|u_0\|_p \neq \widetilde{\lambda}_1$. Moreover, since $K(u_0) \leq 0$, by (75) we have $B_1^{-2} \|u_0\|_p^2 \leq \|\nabla u_0\|_2^2 \leq \|u_0\|_p^p$ and so $\|u_0\|_p \geq B_1^{-p/(p-2)} = \widetilde{\lambda}_1$, concluding the proof. \square

In what follows we shall use the following derivation formula, which is proved here for the sake of completeness only.

Lemma 3. *Under the assumptions of Theorem 1, let u be a weak solution of problem (2) in $[0, T] \times \Omega$. Then*

$$(81) \quad \frac{d}{dt} \|u(t)\|_p^p = p \int_{\Omega} |u(t)|^{p-2} u(t) u_t(t) \quad \text{for almost all } t \in (0, T).$$

Proof. By Definition 1– (a) and (3) we have $|u|^p \in L^\infty(0, T; L^1(\Omega))$, and consequently $\int_{\Omega} |u|^p \in L^\infty(0, T) \subset L^2(0, T)$. It also follows that $u \in H^1((0, T) \times \Omega)$. Since the real function $x \mapsto |x|^p$ is locally Lipschitz continuous, by the chain rule in Sobolev spaces (see [44]) the function $t \mapsto |u(t, x)|^p$ is absolutely continuous for almost all $x \in \Omega$ and $\frac{\partial}{\partial t} |u|^p = p|u|^{p-2} u u_t \in L^2(0, T; L^1(\Omega)) \hookrightarrow L^1((0, T) \times \Omega)$, where assumption (3) was used again. It follows that for all $\varphi \in C_c^\infty(\Omega)$ and $\chi \in C_c^\infty(0, T)$ we have $\int_{(0, T) \times \Omega} |u|^p \varphi \chi' = - \int_{(0, T) \times \Omega} p|u|^{p-2} u u_t \varphi \chi$. Using Fubini's Theorem, since φ is arbitrary it follows that $\int_0^T |u|^p \chi' = - \int_0^T p|u|^{p-2} u u_t \chi$ in $L^1(\Omega)$. Since $\int_{\Omega} p|u|^{p-2} u u_t \in L^2(0, T)$ it follows from last formula that $\|u\|_p^p \in H^1(0, T)$ and (81) holds in the weak sense. By [9, Theorem 8.2] we see that it holds also almost everywhere in $(0, T)$, concluding the proof. \square

We now show that W_s and W_u are invariant under the flow generated by (2).

Lemma 4. *Under the assumptions of Theorem 1, let u be the weak maximal solution of problem (2). Also assume that (9) holds. Then*

- (i) if $u_0 \in W_s$ we have $u(t) \in W_s$ for all $t \in [0, T_{\max})$;
- (ii) if $u_0 \in W_u$ we have $u(t) \in W_u$ for all $t \in [0, T_{\max})$.

Proof. By Lemma 3, the energy identity (6) can be written as

$$(82) \quad J(u(\tau)) \Big|_s^t = - \int_s^t (\|u_t(\tau)\|_{m, \Gamma_1}^m + \|u_t(\tau)\|_2^2) d\tau.$$

Consequently $t \mapsto J(u(t))$ is decreasing in $[0, T_{\max})$ and by Lemma 2

$$(83) \quad J(u(t)) \leq J(u_0) < E_1 \quad \text{for all } t \in [0, T_{\max}).$$

On the other hand, by (75) we have the inequality $J(u(t)) \geq g(\|\nabla u(t)\|_2)$, where $g(\lambda) = \lambda^2/2 - B_1^p \lambda^p/p$ for $\lambda \geq 0$. It is straightforward to verify that g is increasing in $[0, \lambda_1)$ and decreasing in $[\lambda_1, \infty)$, so λ_1 is the maximum point for g , and that $g(\lambda_1) = E_1$. Consequently, by (83) we have $\|\nabla u(t)\|_2 \neq \lambda_1$ for all $t \in [0, T_{\max})$. Since the function $t \mapsto \|\nabla u(t)\|_2$ is continuous, by Lemma 2 the proof is complete. \square

Proof of Theorem 3. By Theorem 2 we just have to prove that when $u_0 \in W_s$ the alternative (8) in Theorem 2 leads to a contradiction, which is obtained by combining Lemma 4–(i) with the Poincarè type inequality recalled at the beginning of the section. \square

Proof of Theorem 4. By Theorem 2 it is enough to prove that there are no solutions in the whole $(0, \infty) \times \Omega$. We argue by contradiction. Since $J(u_0) < E_1$, we can fix $E_2 \in (J(u_0), E_1)$. We set

$$(84) \quad H(t) := E_2 - J(u(t)).$$

By using (77), Lemma 2 and Lemma 4 we get

$$(85) \quad H(t) < E_1 - \frac{1}{2} \|\nabla u(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p \leq E_1 - \frac{1}{2} \lambda_1^2 + \frac{1}{p} \|u(t)\|_p^p \leq \frac{1}{p} \|u(t)\|_p^p.$$

By (82) we have

$$(86) \quad H'(t) = \|u_t(t)\|_{m,\Gamma_1}^m + \|u_t(t)\|_2^2 \geq 0,$$

so that

$$(87) \quad H(t) \geq H(0) = E_2 - J(u_0) > 0.$$

Since, as claimed in Remark 3, it is trivial to verify that $m_0(p) \leq p$ for $p \geq 2$, by (3) and (14) we have $m < 1 + 2^*/2$, which is nothing but the Sobolev critical exponent for the trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ (see [1, Theorem 5.22, p. 114]). Hence we have that $u(t)|_{\Gamma_1} \in L^m(\Gamma_1)$ for all $t \in [0, T_{\max})$, so we can take $\phi = u(t)$ in (42). In this way (here and in the sequel of the proof explicit dependence on t will be omitted) we obtain the identity

$$(88) \quad \|u\|_p^p - \|\nabla u\|_2^2 = \int_{\Gamma_1} |u_t|^{m-2} u_t u + (u_t, u).$$

We estimate the two terms in right-hand side of (88) separately. By Hölder inequality we get

$$(89) \quad \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_{m,\Gamma_1}.$$

To estimate the $L^m(\Gamma_1)$ norm of $u|_{\Gamma_1}$ we first recall the trace embedding for Sobolev space of fractional order (see [1, Theorem 7.58, p. 218] and [55]) $H^s(\mathbb{R}^n) \hookrightarrow W^{\chi,l}(\mathbb{R}^{n-1})$ when $2 \leq l < \infty$, $\chi = s - \frac{n}{2} + \frac{n-1}{l} > 0$. Since $W^{\chi,l}(\mathbb{R}^{n-1}) \hookrightarrow L^l(\mathbb{R}^{n-1})$, using the C^1 regularity of Ω and a standard partition of the unity we have the trace embedding $H^s(\Omega) \hookrightarrow L^l(\partial\Omega)$ when $2 \leq l < \infty$, $s - \frac{n}{2} + \frac{n-1}{l} > 0$ and $0 < s \leq 1$. Using the last embedding with $l = \max\{2, m\}$, the fact that $\partial\Omega$ has finite surface measure and Hölder inequality we get

$$(90) \quad \|u\|_{m,\Gamma_1} \leq C_1 \|u\|_{H^s(\Omega)}$$

with $C_1 = C_1(m, s, \Omega) > 0$, when

$$(91) \quad \max \left\{ \frac{1}{2}, \frac{n}{2} - \frac{n-1}{m} \right\} < s < 1.$$

Next, by the interpolation inequality (see [40, p.49]³) and the already quoted Poincaré type inequality, we have

$$(92) \quad \|u\|_{H^s(\Omega)} \leq C_2 \|u\|_2^{1-s} \|\nabla u\|_2^s$$

$C_2 = C_2(s, \Omega, \Gamma_0) > 0$. By combining (90) and (92) we get

$$(93) \quad \|u\|_{m,\Gamma_1} \leq C_3 \|u\|_2^{1-s} \|\nabla u\|_2^s$$

³Actually interpolation inequality is stated in the quoted reference only for C^∞ domains Ω , but as explicitly remarked there this assumption is not optimal. In particular, since $0 < s \leq 1$, the C^1 regularity assumed here is sufficient to prove the result. Unfortunately the authors were not able to find a reference where interpolation inequality is stated under optimal regularity assumptions.

for some $C_3 = C_3(m, s, \Omega, \Gamma_0) > 0$. By (89) and (93)

$$(94) \quad \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \|u_t\|_{\Gamma_1, m}^{m-1} \|u\|_2^{1-s} \|\nabla u\|_2^s.$$

By weighted Young inequality, if

$$(95) \quad s < \frac{2}{m},$$

for any $\delta > 0$ we have the estimate

$$\left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \left[C_4(\delta) \|u_t\|_{m, \Gamma_1}^m + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p \right] \|u\|_p^{1-s-p(1/m-s/2)}$$

where $C_4(\delta) = C_4(\delta, m, s) > 0$. Consequently, if $1 - s - p(\frac{1}{m} - \frac{s}{2}) < 0$, that is if

$$(96) \quad s < \left(\frac{p}{m} - 1 \right) / \left(\frac{p}{2} - 1 \right),$$

setting $\bar{\alpha}_s = -[1 - s - p(1/m - s/2)]/p > 0$ we obtain

$$(97) \quad \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| \leq C_3 \left[C_4(\delta) \|u_t\|_{m, \Gamma_1}^m + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p \right] \|u\|_p^{-p\bar{\alpha}_s}.$$

Now we have to show the existence of a value of the parameter s satisfying (91), (95) and (96). When $1 < m \leq 2$ we have $\frac{p}{m} - 1 > \frac{p}{2} - 1$ and $\frac{2}{m} > 1$, so (91), (95) and (96) reduce to $\frac{1}{2} < s < 1$. When $m > 2$ we have $(\frac{p}{m} - 1) / (\frac{p}{2} - 1) \leq \frac{2}{m} \leq 1$ and $\frac{n}{2} - \frac{n-1}{m} > \frac{1}{2}$, so (91), (95) and (96) reduce to $\frac{n}{2} - \frac{n-1}{m} < s < (\frac{p}{m} - 1) / (\frac{p}{2} - 1)$. Clearly such an s does exist by assumption (14). We fix it.

Now we consider the second term in the right hand side of (88). Since $p > 2$ and Ω is bounded, applying Hölder inequality we easily get

$$|(u_t, u)| \leq \|u_t\|_2 \|u\|_2 \leq C_5 \|u_t\|_2 \|u\|_p = C_5 \|u_t\|_2 \|u\|_p^{\frac{p}{2}} \|u\|_p^{1-\frac{p}{2}},$$

where $C_5 = C_5(\Omega, p) > 0$. By weighted Young inequality, for any $\delta > 0$ we obtain

$$(98) \quad \left| \int_{\Omega} u_t u \right| \leq C_5 \left[\frac{1}{4\delta} \|u_t\|_2^2 + \delta \|u\|_p^p \right] \|u\|_p^{1-\frac{p}{2}}.$$

Now we set

$$(99) \quad \bar{\beta}_s = \min \left\{ \bar{\alpha}_s, -\frac{1}{p} + \frac{1}{2} \right\}.$$

Since $p > 2$ we have $\bar{\beta}_s > 0$. Since, by (85) and (87) we have

$$(100) \quad \|u\|_p \geq [pH(0)]^{1/p},$$

we can combine (97) and (98) (by also using (99)) to obtain

$$(101) \quad \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u \right| + |(u_t, u)| \leq C_7 \left[C_6(\delta) \left(\|u_t\|_{m, \Gamma_1}^m + \|u_t\|_2^2 \right) + \delta \|\nabla u\|_2^2 + 2\delta \|u\|_p^p \right] \|u\|_p^{-p\bar{\beta}_s},$$

where $C_7 = C_7(m, p, \Omega, H(0)) > 0$ and $C_6(\delta) = C_6(\delta, p, m) > 0$. By combining (88) with (101) we get

$$(102) \quad \|u\|_p^p - \|\nabla u\|_2^2 \leq C_7 \left[C_6(\delta) \left(\|u_t\|_{m, \Gamma_1}^m + \|u_t\|_2^2 \right) + \delta \|\nabla u\|_2^2 + 2\delta \|u\|_p^p \right] \|u\|_p^{-p\bar{\beta}_s}$$

Consequently, by (86) and (100)

$$\begin{aligned} \|u\|_p^p - \|\nabla u\|_2^2 &\leq C_8(\delta)H'(t) \|u\|_p^{-p\bar{\beta}_s} + C_7(pH(0))^{-\bar{\beta}_s}\delta [\|\nabla u\|_2^2 + 2\|u\|_p^p] \\ &\leq C_8(\delta)H'(t) \|u\|_p^{-p\bar{\beta}_s} + C_9\delta [\|\nabla u\|_2^2 + \|u\|_p^p] \end{aligned}$$

where $C_8(\delta) = C_8(\delta, m, p, H(0), \Omega) > 0$ and $C_9 = C_9(m, p, H(0), \Omega) > 0$. Consequently

$$2(1 + C_9\delta) \left[-\frac{\|\nabla u\|_2^2}{2} \right] + p(1 - C_9\delta) \frac{1}{p} \|u\|_p^p \leq C_8(\delta)H'(t) \|u\|_p^{-p\bar{\beta}_s}.$$

By (84) the last estimate can be rewritten as

$$\begin{aligned} (103) \quad &2(1 + C_9\delta)H(t) - 2(1 + C_9\delta)E_2 + [p(1 - C_9\delta) - 2(1 + C_9\delta)] \frac{1}{p} \|u\|_p^p \\ &\leq C_8(\delta)H'(t) \|u\|_p^{-p\bar{\beta}_s}. \end{aligned}$$

Now, by Lemma 4, (76) and (80) we have $\|u\|_p^p \geq \widetilde{\lambda}_1^p = \lambda_1^2$, so previous estimates yields

$$\begin{aligned} (104) \quad &2(1 + C_9\delta)H(t) - 2(1 + C_9\delta)E_2 + \lambda_1^2 \left[(1 - C_9\delta) - \frac{2}{p}(1 + C_9\delta) \right] \\ &\leq C_8(\delta)H'(t) \|u\|_p^{-p\bar{\beta}_s}. \end{aligned}$$

Now, since $E_2 < E_1$, using (76) and the fact that C_9 is independent on δ , as $\delta \rightarrow 0^+$ we have

$$\begin{aligned} -2(1 + C_9\delta)E_2 + \lambda_1^2 \left[(1 - C_9\delta) - \frac{2}{p}(1 + C_9\delta) \right] &\rightarrow -2E_2 + \lambda_1^2 \frac{p-2}{p} \\ &> -2E_1 + \lambda_1^2 \frac{p-2}{p} = 0 \end{aligned}$$

Hence, by fixing $\delta > 0$ sufficiently small, there exists two positive constants C_{10} and C_{11} dependent on $m, p, H(0)$ and Ω such that

$$(105) \quad C_{10}H(t) \leq C_{11}H'(t) \|u\|_p^{-p\bar{\beta}_s}.$$

By (85) the last estimate implies that

$$(106) \quad H'(t) \geq C_{12}H_p^{1+\bar{\beta}_s}(t)$$

where $C_{12} = C_{12}(m, p, H(0), \Omega) > 0$, which by integration yields the required contradiction, concluding the proof. \square

5. MORE GENERAL RESULTS

This section is devoted to generalize our results to problem (1), where Q and f satisfy suitable assumptions which generalize the specific behaviour of $|u_t|^{m-2}u_t$ and $|u|^{p-2}u$. Our assumptions on Q are the following ones.

- (Q1) Q is a Carathéodory real function defined on $(0, \Theta) \times \Gamma_1 \times \mathbb{R}$ for some $\Theta > 0$, $Q(t, x, 0) = 0$ for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$, and there exist an exponent

$m > 1$ and positive constants c_1, c_2, c_3 and c_4 , possibly dependent on Θ , such that

$$\begin{aligned} c_1 |v|^{m-1} &\leq |Q(t, x, v)| \leq c_2 |v|^{m-1} \quad \text{when } |v| \geq 1 \quad \text{and} \\ c_3 |v|^{m-1} &\leq |Q(t, x, v)| \leq c_4 \quad \text{when } |v| \leq 1 \end{aligned}$$

for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$ and all $v \in \mathbb{R}$.

(Q2) The function $Q(t, x, \cdot)$ is increasing for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$.

Remark 6. When $Q = Q(v)$ assumptions (Q1)–(Q2) reduce (independently on Θ) to assume that $Q \in C(\mathbb{R})$ is increasing and such that

$$Q(0) = 0, \quad \lim_{v \rightarrow 0} \frac{|Q(v)|}{|v|^{m-1}} > 0, \quad 0 < \lim_{|v| \rightarrow \infty} \frac{|Q(v)|}{|v|^{m-1}} \leq \overline{\lim}_{|v| \rightarrow \infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty,$$

as for example $Q = Q_0(v) = a|v|^{\mu-2}v + b|v|^{m-2}v$, $a \geq 0$, $b > 0$, $1 < \mu \leq m$. Moreover (Q1–2) are also satisfied for any $\Theta > 0$ by $Q = Q_1(t, v) = d(t)Q_0(v)$, where $d \in L_{\text{loc}}^\infty([0, \infty))$, $d > 0$, $1/d \in L_{\text{loc}}^\infty([0, \infty))$.

Remark 7. Let us note, for a future use, that (Q1)–(Q2) yield the existence of positive constants c_5 and c_6 (possibly dependent on Θ) such that

$$(107) \quad |Q(t, x, v)| \leq c_5(1 + |v|^{m-1})$$

and

$$(108) \quad Q(t, x, v)v \geq c_6 |v|^m$$

for almost all $(t, x) \in (0, \Theta) \times \Gamma_1$ and all $v \in \mathbb{R}$.

5.1. Forced heat equation. We first present our generalization of Theorem 5 to the problem

$$(109) \quad \begin{cases} u_t - \Delta u = g(t, x) & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -Q(t, x, u_t) & \text{on } [0, T] \times \Gamma_1, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

where g is a given term acting on Ω and $T > 0$ is fixed.

Definition 3. Let $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $g \in L^2((0, T) \times \Omega)$. We say that u is a weak solution of (109) in $[0, T] \times \Omega$ if (a–d) of Definition 1 hold, with the distribution identity (21) being replaced by

$$(110) \quad \int_{\Omega} u_t(t)\phi + \int_{\Omega} \nabla u(t) \nabla \phi + \int_{\Gamma_1} Q(t, \cdot, u_t(t)) u_t(t)\phi = \int_{\Omega} g(t)\phi,$$

which makes sense due to (107).

Theorem 6. Suppose that (Q1) and (Q2) hold with $\Theta = T$ and that $g \in L^2((0, T) \times \Omega)$. Then, given any initial datum $u_0 \in H_{\Gamma_0}^1(\Omega)$, there is a unique weak solution u of (109) in $[0, T] \times \Omega$. Moreover (16) and (17) hold and u satisfies the energy identity

$$\frac{1}{2} \|\nabla u\|_2^2 \Big|_s^t + \int_s^t \|u_t\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t) u_t = \int_s^t \int_{\Omega} g u_t$$

for $0 \leq s \leq t \leq T$. Finally, given any couple of initial data $u_{01}, u_{02} \in H_{\Gamma_0}^1(\Omega)$ and any couple of forcing terms $g_1, g_2 \in L^2((0, T) \times \Omega)$, denoting by u^1 and u^2 the

solutions of (109) respectively corresponding to u_{01} , g_1 and to u_{02} , g_2 , the estimate (19) holds true.

Sketch of the proof of Theorem 6. Using (107), (108) and (Q2) we can repeat almost verbatim the proof of Theorem 5 by replacing everywhere $|u_t^k|^{m-2}u_t^k$ with $Q(t, x, u_t^k)$, so starting from the problem

$$(111) \quad \begin{cases} (u_t^k, w_j) + (\nabla u_t^k, \nabla w_j) + \int_{\Gamma_1} Q(\cdot, \cdot, u_t^k) w_j = \int_{\Omega} g w_j, & j = 1, \dots, k, \\ u^k(0) = u_{0k}. \end{cases}$$

The definition (33) is now replaced by $G_k(t, y) = y + \int_{\Gamma_1} Q(t, \cdot, B_k(x) \cdot y) B_k(x) dx$, $t \in (0, T)$, $y \in \mathbb{R}^k$, so in the generalization of problem (34) now G_k explicitly depends on t . By using assumption (Q2) the arguments of [59, Proof of Theorem 1.5] continue to work in this more general situation for any fixed $t \in (0, T)$, while all the other estimates keep unchanged. The energy identity (36) continues to hold provided the term $\|u_t^k\|_{m, \Gamma_1}$ is replaced by the term $\int_{\Gamma_1} Q(t, x, u_t^k) u_t^k$. By using (108) and (Q2) we still get (37) with $|u_t^k|^{m-2}u_t^k$ being replaced by $Q(t, x, u_t^k)$ in the forth line, where now C' depends also on $c_1 - c_4$. Finally, to apply the monotonicity method we use (Q2), which is also used in the proof of estimate (19). \square

5.2. Local well-posedness. We generalize Theorem 1 to problem (1) under the following assumption on f :

- (F1) f is a Carathéodory real function defined on $\Omega \times \mathbb{R}$, $f(x, 0) = 0$ for almost all $x \in \Omega$ and there is an exponent $p \geq 2$ and a positive constant c_7 such that for almost all $x \in \Omega$ and all $u_1, u_2 \in \mathbb{R}$

$$|f(x, u_1) - f(x, u_2)| \leq c_7 |u_1 - u_2| (1 + |u_1|^{p-2} + |u_2|^{p-2}).$$

An explicit example of a function f which satisfies (F1) (use (55)) is given by

$$(112) \quad f = f_0(x, u) = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, \quad 2 \leq q \leq p, \quad a, b \in L^\infty(\Omega).$$

When f is independent on x assumption (F1) can be equivalently written as follows:

$$(113) \quad f \in W_{loc}^{1, \infty}(\mathbb{R}), \quad f(0) = 0, \quad |f'(u)| = O(|u|^{p-2}) \text{ as } |u| \rightarrow \infty.$$

A further example of a non-algebraic nonlinearity satisfying (113) is given by $f = \pm f_1$, where $f_1(u) = |u|^{p-2}u$, $p \geq 2$, when $|u| \geq 1$ while $f_1(u) = u$ when $|u| \leq 1$.

Remark 8. We note that an immediate consequence of (F1) is the existence of a positive constant c_8 such that

$$(114) \quad |f(x, u)| \leq c_8(|u| + |u|^{p-1})$$

for almost all $x \in \Omega$ and all $u \in \mathbb{R}$.

Definition 4. Let $u_0 \in H_{\Gamma_0}^1(\Omega)$ and suppose that (Q1-2), (F1) and assumption (3) hold. We say that u is a weak solution of (1) in $[0, T] \times \Omega$ if (a-d) of Definition 1 hold, with the distribution identity (21) being replaced by

$$(115) \quad \int_{\Omega} u_t(t) \phi + \int_{\Omega} \nabla u(t) \nabla \phi + \int_{\Gamma_1} Q(t, x, u_t(t)) \phi = \int_{\Omega} f(x, u(t)) \phi.$$

Moreover we say that u is a weak solution of problem (1) in $[0, T) \times \Omega$ if it is a weak solution of (1) in $[0, T'] \times \Omega$ for all $T' \in (0, T)$.

Theorem 1 is generalized as follows.

Theorem 7. *Suppose that (Q1), (Q2) and (F1) hold together with (3). Then given any initial datum $u_0 \in H_{\Gamma_0}^1(\Omega)$ there is $T^* = T^*(\|u_0\|_{H_{\Gamma_0}^1}, m, p, \Omega, \Gamma_1, c_1, c_2, c_3, c_4)$ in $(0, \min\{1, \Theta\}]$, decreasing in the first variable, such that problem (1) has a unique weak solution in $[0, T^*] \times \Omega$. Moreover (4), (5) and (7) hold, together with the energy identity*

$$(116) \quad \frac{1}{2} \|\nabla u\|_2^2 \Big|_s^t + \int_s^t \|u_t\|_2^2 + \int_s^t \int_{\Gamma_1} Q(\cdot, \cdot, u_t) u_t = \int_s^t \int_{\Omega} f(\cdot, u) u_t$$

for $0 \leq s \leq t \leq T^*$.

Sketch of the proof. We repeat the proof of Theorem 1. We take $T \leq \Theta$ and $u \in X_T$. We note that, by (114) and (3), we have $f(\cdot, u) \in L^\infty(0, T; L^2(\Omega))$. So by Theorem 6 there is a unique solution v of the problem

$$(117) \quad \begin{cases} v_t - \Delta v = f(x, u) & \text{in } (0, T^*) \times \Omega, \\ v = 0 & \text{on } [0, T^*) \times \Gamma_0, \\ \frac{\partial v}{\partial \nu} = -Q(t, x, v_t) & \text{on } [0, T^*) \times \Gamma_1, \\ v(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

We set $\Phi : X_T \rightarrow X_T$ by $\Phi(u) = v$. By using the same arguments in the proof of Theorem 1 together with assumptions (Q2) and (F1) we get for any $u \in B_R$, the estimate

$$(118) \quad \frac{1}{2} \|\nabla v(t)\|_2^2 + \frac{1}{2} \int_0^t \|v_t\|_2^2 \leq \frac{1}{2} \|\nabla u_0\|_2^2 + K_2(R^2 + R^{2(p-1)})T$$

which generalizes (47) to this more general situation, where now the constants K_i depends also on c_7 . Then we proceed as in the quoted proof with $(R^2 + R^{2(p-1)})$ replacing $R^{2(p-1)}$. Consequently we get that $\Phi(B_R) \subset B_R$ provided that

$$(119) \quad R = 4R_0, \quad \text{and} \quad T \leq \min \left\{ 1, \Theta, K_3(16 + 16^{p-1}R_0^{2(p-2)})^{-1} \right\},$$

generalizing (52). In order to show that, for suitable T , Φ is a contraction in B_R we proceed exactly as in the quoted proof by taking $u, \bar{u} \in B_R$, $v = \Phi(u)$, $\bar{v} = \Phi(\bar{u})$, $w = v - \bar{v}$. Clearly, w is a weak solution of the problem

$$(120) \quad \begin{cases} w_t - \Delta w = f(x, u) - f(x, \bar{u}) & \text{in } (0, T) \times \Omega, \\ w = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial w}{\partial \nu} = -Q(t, x, v_t) + Q(t, x, \bar{v}_t) & \text{on } [0, T) \times \Gamma_1, \\ w(0, x) = 0 & \text{in } \Omega. \end{cases}$$

generalizing (53). Since by (114) we have $f(\cdot, u), f(\cdot, \bar{u}) \in L^\infty(0, T; L^2(\Omega))$ and by (107) we have $Q(\cdot, \cdot, v_t), Q(\cdot, \cdot, \bar{v}_t) \in L^{m'}((0, T) \times \Gamma_1)$ we can apply Lemma 1 to get

$$\begin{aligned} \frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 + \int_0^t \int_{\Gamma_1} [Q(\cdot, \cdot, v_t) - Q(\cdot, \cdot, \bar{v}_t)] w_t \\ = \int_0^t \int_{\Omega} [f(\cdot, u) - f(\cdot, \bar{u})] w_t. \end{aligned}$$

Using (F1) and (Q2) we generalize the estimate (56) to the following one

$$(121) \quad \frac{1}{2} \|\nabla w(t)\|_2^2 + \int_0^t \|w_t\|_2^2 \leq c_7 \int_0^T \int_{\Omega} |u - \bar{u}| (1 + |u|^{p-2} + |\bar{u}|^{p-2}) |w_t|.$$

Consequently exactly the same arguments used in the quoted proof allow to prove the estimate

$$(122) \quad \|w(t)\|_{H_{\Gamma_0}^1}^2 \leq K_{13}^2 (1 + R^{(p-2)})^2 T \|u - \bar{u}\|_{L^\infty(0,T;H_{\Gamma_0}^1(\Omega))}^2.$$

replacing (63), so by (119)₁, Φ is a contraction provided $T < K_{13}^{-2} (1 + 4^{p-2} R_0^{p-2})^{-2}$. We can finally fix T^* and complete the proof. \square

The following result is nothing but the generalization of Theorem 2.

Theorem 8. *Suppose that (Q1-2) hold for all $\Theta > 0$, together with (F1) and (3). Then the assertions of Theorem 2 hold when problem (2) is replaced by (1).*

Sketch of the proof. We describe the adaptations needed to cover this more general situation with respect to the arguments used in the proof of Theorem 2. The existence of a unique weak maximal solution of (1) follows exactly in the same way. When proving the alternative (i-ii), since the equation is not autonomous (as the term Q is explicitly time-dependent) a more detailed explanation is needed. Let us suppose by contradiction that (66) holds, so there is a sequence $T_n \rightarrow T_{\max}^- < \infty$ such that $\|u(T_n)\|_{H_{\Gamma_0}^1}$ is bounded. Since Q satisfies assumptions (Q1-2) for all positive Θ , we can choose $\Theta = T_{\max} + 1$. We set for any $n \in \mathbb{N}$ the time-shifted nonlinear term $Q_n(t, x, v) = Q(t + T_n, x, v)$, which satisfies assumptions (Q1-2) with $\Theta = \Theta_n := T_{\max} - T_n + 1 \geq 1$, so that Q_n satisfies the same assumptions for $\Theta = 1$ for all $n \in \mathbb{N}$. It follows that the existence time T^* assured by Theorem 7 is independent on n , so problem (1) with initial time T_n and initial datum $u(T_n)$ has a unique weak solution in $[T_n, T_n + T^*] \times \Omega$, which leads to the desired contradiction.

When proving the continuous dependence of the solution u on the initial datum we get the energy identity

$$\begin{aligned} \frac{1}{2} \|\nabla w^n\|_2^2 \Big|_0^t + \int_0^t \|w_t^n\|_2^2 + \int_0^t \int_{\Gamma_1} [Q(\cdot, \cdot, u_t^n) - Q(\cdot, \cdot, u_t)] w_t^n \\ = \int_0^t \int_{\Omega} [f(\cdot, u^n) - f(\cdot, u)] w_t^n \end{aligned}$$

generalizing (68). Using assumptions (Q2), (F1) and (3) we then get the estimate (69) again, so we can conclude the proof exactly as in Theorem 2. \square

5.3. Global existence versus blow-up. In order to generalize Theorems 3 and 4 to problem (1) we first generalize Lemma 3. We introduce the notation

$$F(x, u) = \int_0^u f(x, s) ds.$$

Lemma 5. *Under the assumptions of Theorem 7, let u be a weak solution of problem (1) in $[0, T] \times \Omega$. Then*

$$(123) \quad \frac{d}{dt} F(\cdot, u(t)) = \int_{\Omega} f(\cdot, u(t)) u_t(t) \quad \text{for almost all } t \in (0, T).$$

Proof. We first note that an immediate consequence of (114) is that $|F(x, u)| \leq c_9(1 + |u|^p)$ for a positive constant c_9 . Hence $\int_{\Omega} F(\cdot, u) \in L^\infty(0, T) \subset L^2(0, T)$. Consequently exactly the same arguments used in the proof of Lemma 3 apply to this more general case. \square

To extend in a suitable way the definition of the stable and unstable sets we need to introduce a second structural assumption on the nonlinearity f .

(F2) There is $c_{10} \geq 0$ such that $F(x, u) \leq \frac{c_{10}}{p} |u|^p$ for almost all $x \in \Omega$ and all $u \in \mathbb{R}$.

We remark that the model nonlinearity f_0 defined in (112) satisfies (F2) if and only if $a \leq 0$.

When $\sigma(\Gamma_0) > 0$, $p > 2$ and (F2) holds we set

$$(124) \quad D_1 := \sup_{u \in H_{\Gamma_0}^1(\Omega), u \neq 0} \frac{\int_{\Omega} F(\cdot, u)}{\|\nabla u\|_2^p} \leq \frac{c_{10}}{p} B_1^p.$$

When $D_1 > 0$ we also set

$$(125) \quad \lambda_1 = (pD_1)^{-1/(p-2)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2,$$

while $\lambda_1 = E_1 = +\infty$ when $D_1 \leq 0$. Moreover we denote

$$(126) \quad J(u) = \frac{1}{2} \|\nabla u_0\|_2^2 - \int_{\Omega} F(\cdot, u) \quad \text{for any } u \in H_{\Gamma_0}^1(\Omega),$$

$$(127) \quad W_s = \{u_0 \in H_{\Gamma_0}^1(\Omega) : \|\nabla u_0\| < \lambda_1 \text{ and } J(u_0) < E_1\},$$

$$(128) \quad W_u = \{u_0 \in H_{\Gamma_0}^1(\Omega) : \|\nabla u_0\| > \lambda_1 \text{ and } J(u_0) < E_1\}.$$

Clearly due to Lemma 2 when $f = |u|^{p-2}u$ definitions (127) and (128) coincide with (12) and (13), even if they are inspired from (77) and (79).

We now generalize the potential-well argument contained in Lemma 4.

Lemma 6. *Suppose that (Q1-2) hold for all $\Theta > 0$, together with (F1-2) and (3). Suppose moreover that $\sigma(\Gamma_0) > 0$ and $p > 2$. Then the conclusion of Lemma 4 continue to hold.*

Proof. By Lemma 5 the energy identity (116) can be written as

$$(129) \quad J(u(\tau)) \Big|_s^t = - \int_s^t \int_{\Gamma_1} Q(\tau, \cdot, u_t(\tau)) u_t(\tau) d\tau - \int_s^t \|u_t\|_2^2 d\tau.$$

By (129) and (Q2) the energy function $E(t) := J(u(t))$ is decreasing in $[0, T_{\max})$. Hence (83) continue to hold. By (124) we have $J(u(t)) \geq \tilde{g}(\|\nabla u(t)\|_2)$, where $\tilde{g} = \frac{\lambda^2}{2} - D_1 \lambda^p$ if $D_1 > 0$, while $\tilde{g} = \frac{\lambda^2}{2}$ if $D_1 \leq 0$. Then, when $D_1 > 0$, the same arguments used in the proof of Lemma 4 apply, while there is nothing to prove when $D_1 \leq 0$. \square

We can now state the generalization of Theorem 3.

Theorem 9. *Under the assumptions of Lemma 6 if $u_0 \in W_s$ then $T_{\max} = \infty$ and $u(t) \in W_s$ for all $t \geq 0$.*

Proof. When $D_1 > 0$ we can exactly repeat the proof of Theorem 3 by using Lemma 6. When $D_1 \leq 0$ the same argument applies since in this case we have $J(u) \geq \frac{1}{2} \|\nabla u\|_2^2$ so W_s is bounded. \square

In order to generalize Theorem 4 we need to strengthen assumption (Q1-2) to the following ones.

(Q1') Q is a Carathéodory real function defined on $(0, \infty) \times \Gamma_1 \times \mathbb{R}$, $Q(t, x, 0) = 0$ for almost all $(t, x) \in (0, \infty) \times \Gamma_1$, and there exists exponents $1 < \mu \leq m$, a positive function d such that $d, 1/d \in L_{\text{loc}}^\infty([0, \infty))$, and positive constants c'_1, c'_2, c'_3 and c'_4 such that

$$\begin{aligned} c'_1 d(t) |v|^{m-1} &\leq |Q(t, x, v)| \leq c'_2 d(t) |v|^{m-1} \quad \text{when } |v| \geq 1, \quad \text{and} \\ c'_3 d(t) |v|^{\mu-1} &\leq |Q(t, x, v)| \leq c'_4 d(t) |v|^{\mu-1} \quad \text{when } |v| \leq 1 \end{aligned}$$

for almost all $(t, x) \in (0, \infty) \times \Gamma_1$ and all $v \in \mathbb{R}$.

(Q2') The function $Q(t, x, \cdot)$ is increasing for almost all $(t, x) \in (0, \infty) \times \Gamma_1$.

Remark 9. We remark that the nonlinearities Q_0 and Q_1 defined in Remark 6 satisfy as well assumption (Q1'-2'). Moreover when $Q = Q(v)$ these assumptions reduce to assume that $Q \in C(\mathbb{R})$ is increasing and

$$0 < \lim_{v \rightarrow 0} \frac{|Q(v)|}{|v|^{\mu-1}} \leq \overline{\lim}_{v \rightarrow 0} \frac{|Q(v)|}{|v|^{\mu-1}} < \infty, \quad 0 < \lim_{|v| \rightarrow \infty} \frac{|Q(v)|}{|v|^{m-1}} \leq \overline{\lim}_{|v| \rightarrow \infty} \frac{|Q(v)|}{|v|^{m-1}} < \infty.$$

We also note, for future use, some further consequences of (Q1') and (Q2'). Since $Q(0) = 0$, by (Q2) we have $Q(t, x, v)v \geq 0$, so $Q(t, x, v)v = |Q(t, x, v)||v|$. Hence, when $|v| \geq 1$ we have

$$(130) \quad |Q(t, x, v)| \leq c'_5 d^{1/m}(t) [Q(t, x, v)v]^{1/m'}$$

while when $|v| \leq 1$

$$(131) \quad |Q(t, x, v)| \leq c'_6 d^{1/\mu}(t) [Q(t, x, v)v]^{1/\mu'}$$

for almost all $(t, x) \in (0, \infty) \times \Gamma_1$, where c'_5 and c'_6 are positive constants.

In order to state our blow-up result for problem (1) we need a further specific structural assumption on f .

(F3) There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there exists $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$f(x, u)u - (p - \varepsilon)F(x, u) \geq c_{11} |u|^p$$

for almost all $x \in \Omega$ and all $u \in \mathbb{R}$.

Clearly the model nonlinearity f_0 defined in (112) satisfies (F2-3) if and only if $a \leq 0$. We can finally state

Theorem 10. *Suppose that (Q1'-Q2'), (F1-3), (3) and (14) hold. Moreover suppose that $\sigma(\Gamma_0) > 0$, $p > 2$,*

$$(132) \quad \int_0^\infty \frac{dt}{d^{1/(m-1)} + d^{1/(\mu-1)}} = \infty$$

and $u_0 \in W_u$. Then the conclusions of Theorem 4 hold.

Remark 10. Assumption (132) needs some comment, as it express the possible time-behavior of Q . When $d(t) = (1+t)^\beta$, $\beta \in \mathbb{R}$, it reduces to $\beta \leq \mu - 1$, and in particular when $\mu = m$ in assumption (Q1') (what happens for example when $Q(v) = d(t)|v|^{m-2}v$), it reduces to $\beta \leq m - 1$, which is a well-known optimal assumption to prevent over-damping for time dependent damping terms in ordinary differential systems.

Proof. As in the proof of Theorem 4 we prove, by contradiction, that there are no solutions in the whole $(0, \infty) \times \Omega$. We fix $E_2 \in (J(u_0), E_1)$ and set H by (84). By using Lemma 6 and (124) we get a slightly generalized version of (85), that is

$$(133) \quad H(t) \leq \frac{c_{10}}{p} \|u(t)\|_p^p.$$

By (Q2') formula (86) is now generalized to

$$(134) \quad H'(t) = \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \|u_t(t)\|_2^2 \geq \int_{\Gamma_1} Q(t, \cdot, u_t) u_t \geq 0$$

so that (87) holds true. By (107) we can again take $\phi = u_t$ in the distribution identity (115) so getting the following generalized version of (88)

$$(135) \quad \int_{\Omega} f(\cdot, u) u - \|\nabla u\|_2^2 = \int_{\Gamma_1} Q(\cdot, \cdot, u_t) u + (u_t, u)$$

The estimate (98) of the second term in the right hand side of (135) keeps unchanged, while the estimate the first term in it needs a more detailed explanation. We use (130), (131) and Hölder inequality twice to get

$$\begin{aligned} I_1 &:= \left| \int_{\Gamma_1} Q(\cdot, \cdot, u_t) u \right| \\ &\leq c'_5 d^{1/m} \int_{\{x \in \Gamma_1 : |u_t| \geq 1\}} [Q(t, \cdot, u_t) u_t]^{1/m'} |u| + c'_6 d^{1/\mu} \int_{\{x \in \Gamma_1 : |u_t| \leq 1\}} [Q(t, \cdot, u_t) u_t]^{1/\mu'} |u| \\ &\leq (c'_5 + c'_6) \left[d^{1/m} \left(\int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/m'} + d^{1/\mu} \left(\int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/\mu'} \right] \|u\|_{m, \Gamma_1} \end{aligned}$$

which generalizes (89). Now we estimate $\|u\|_{m, \Gamma_1}$ in previous formula by using (93). In this way we obtain

$$I_1 \leq C_3 \left[d^{1/m} \left(\int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/m'} + d^{1/\mu} \left(\int_{\Gamma_1} Q(t, \cdot, u_t) u_t \right)^{1/\mu'} \right] \|u\|_2^{1-s} \|\nabla u\|_2^s$$

for exponents s satisfying (91) (generalizing (94)). The same arguments used in the proof of Theorem 4 then give, for any $\delta > 0$,

$$\begin{aligned} I_1 &\leq C_3 \left[C_4(\delta) d^{1/(m-1)} \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right] \|u\|_p^{1-s-p(1/m-s/2)} \\ &\quad + C_3 \left[C_4(\delta) d^{1/(\mu-1)} \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right] \|u\|_p^{1-s-p(1/\mu-s/2)} \end{aligned}$$

provided also (95) (and consequently $s < 2/\mu$ as well) holds. By (87) and (133) we have $\|u\|_p^p \geq \left(\frac{p}{c_{10}} H(0) \right)^{1/p}$, so from previous formula we derive, as $\mu \leq m$,

$$I_1 \leq C'_3 \left\{ C_4(\delta) \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] \int_{\Gamma_1} Q(t, \cdot, u_t) u_t + \delta \|u\|_2^2 + \delta \|u\|_p^p \right\} \|u\|_p^{-p\bar{\alpha}_s},$$

where $C'_3 = C'_3(p, m, s, \Omega, H(0)) > 0$ and $\bar{\alpha}_s = -[1 - s - p(1/m - s/2)]/p > 0$, generalizing (97). By plugging the last estimate and (98) in (135) and using (134) we get

$$\int_{\Omega} f(\cdot, u) u - \|\nabla u\|_2^2 \leq C_8(\delta) \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] H'(t) \|u\|_p^{-p\bar{\beta}_s} + C_9 \delta \left[\|\nabla u\|_2^2 + \|u\|_p^p \right]$$

so generalizing (102). Consequently, by (84) and (126) we have, for any $\varepsilon > 0$,

$$\begin{aligned} & \int_{\Omega} f(\cdot, u)u + (p - \varepsilon)H(t) - (p - \varepsilon)E_2 + \left[\frac{p - \varepsilon}{2} - (1 + C_9\delta) \right] \|\nabla u\|_2^2 \\ & - (p - \varepsilon) \int_{\Omega} F(\cdot, u) - C_9\delta \|u\|_p^p \leq C_8(\delta) \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] H'(t) \|u\|_p^{-p\bar{\beta}_s}, \end{aligned}$$

where $\bar{\beta}_s$ is given by (99). Then, using assumption (F3) for $\varepsilon \in (0, \varepsilon_0]$ we have

$$\begin{aligned} (136) \quad & [c_{11}(\varepsilon) - C_9\delta] \|u\|_p^p + \left[\frac{p - \varepsilon}{2} - (1 + C_9\delta) \right] \|\nabla u\|_2^2 + (p - \varepsilon)H - (p - \varepsilon)E_2 \\ & \leq C_8(\delta) \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] H'(t) \|u\|_p^{-p\bar{\beta}_s}. \end{aligned}$$

By Lemma 6 we have $\|\nabla u\|_2 \geq \lambda_1$, so by (125) we get

$$(137) \quad \left[\frac{p - \varepsilon}{2} - (1 + C_9\delta) \right] \|\nabla u\|_2^2 - (p - \varepsilon)E_2 \geq \left(-C_9\delta - \frac{\varepsilon}{p} \right) \lambda_1^2 + (p - \varepsilon)(E_1 - E_2).$$

We fix $\varepsilon = \varepsilon_1$ small enough in order to have $\frac{\varepsilon}{p} \lambda_1^2 < \frac{p - \varepsilon}{2} (E_1 - E_2)$. After that we fix $\delta = \delta_1$ such that $\frac{C_9\delta}{2} < \frac{p - \varepsilon_1}{2} (E_1 - E_2)$ and $c_{11}(\varepsilon_1) - C_9\delta > 0$. Consequently, from (136) and (137) we obtain

$$(p - \varepsilon_1)H(t) \leq C_8(\delta_1) \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] H'(t) \|u\|_p^{-p\bar{\beta}_s}$$

generalizing (105). By (133) we finally obtain $H'(t) \geq C_9 \left[d^{\frac{1}{m-1}} + d^{\frac{1}{\mu-1}} \right] H^{1+\bar{\beta}_s}(t)$ which generalizes (106). By integrating and using assumption (132) we get the desired contradiction and conclude the proof. \square

APPENDIX A. A PHYSICAL MODEL

This section is devoted to describe a physical model which motivates problem (1). Let Ω represent a solid body surrounded by a fluid denoted by A , with contact Γ_1 and (possibly) having an internal cavity with contact boundary Γ_0 . We suppose that a heat reaction-diffusion process occurs inside Ω such that, if $u = u(t, x)$ represents the temperature at point x and time t , the quantity of heat produced by the reaction is proportional to a superlinear power of the temperature, i.e. to u^{p-1} with $p > 2$. Thus the process can be modelled by the heat equation with source

$$(138) \quad u_t - \rho \Delta u = |u|^{p-2} u \text{ in } (0, T) \times \Omega$$

where the thermal conductivity $\rho > 0$ is taken to be 1 for simplicity. The surrounding fluid is supposed to be a perfect conductor of heat, so the temperature in A is spatially homogeneous and can be described by a number $v = v(t)$ for any $t \geq 0$. In particular, there is no diffusion in the fluid. Such assumption is realistic if the fluid is well stirred. Moreover, we introduce a refrigerating process in the fluid with the help of which one tries to control the reaction inside the solid Ω . We assume that the refrigerating system is controlled in such a way that the heat absorbed from the fluid is proportional to a power of the rate of change of the temperature, as $|v'(t)|^{m-2} v'(t)$. Let $j = j(t, x)$ be the heat flux from Ω to A . Then the rate of change of the temperature $v'(t)$ is given by $v'(t) = -|v'(t)|^{m-2} v(t) + \int_{\Gamma_1} j(t, x) dS$. On the other hand, the heat flux $j(t, x)$ is given by the classical conductivity rule

by $j(t, x) = -\frac{\partial u}{\partial \nu}$, since $\rho = 1$. Finally, the thermal contact of the fluid at Γ_1 yields the continuity condition $u(t, x) = v(t)$, $x \in \Gamma_1$, $t \geq 0$, while the temperature on Γ_0 is assumed to be constant (for simplicity constantly vanishing), that is

$$(139) \quad u(t, x) = 0, \quad x \in \Gamma_0, \quad t \geq 0.$$

Combining (138)-(139), we obtain (1) with $f = |u|^{p-2}u$ and $Q = u_t + |u_t|^{m-2}u_t$. These nonlinear terms are included in theory developed in Section 5. In particular Theorem 10 shows that the refrigerating system cannot avoid the internal explosion with this conditions.

APPENDIX B. GLOBAL EXISTENCE FOR PROBLEM (2) WHEN $p = 2$

This section is devoted to state and prove the global existence result for problem (2) when $p = 2$ mentioned in the Introduction. For the sake of generality we actually shall prove a more general version of it dealing with problem (1).

Theorem 11. *Under the assumptions of Theorem 8 if $p = 2$ then $T_{\max} = \infty$.*

Proof. We suppose by contradiction that $T_{\max} < \infty$. By (116) together with assumptions (Q1-2) and (114) we have

$$\frac{1}{2} \|\nabla u\|_2^2 + \int_0^t \|u_t\|_2^2 \leq \frac{1}{2} \|\nabla u_0\|_2^2 + 2c_8 \int_0^t \int_{\Omega} |u| |u_t|.$$

By Hölder and weighted Young inequalities we consequently get

$$\frac{1}{2} \|\nabla u\|_2^2 + \int_0^t \|u_t\|_2^2 \leq \frac{1}{2} \|\nabla u_0\|_2^2 + 2c_8^2 \int_0^t \|u\|_2^2 + \frac{1}{2} \int_0^t \|u_t\|_2^2$$

and consequently

$$(140) \quad \|\nabla u\|_2^2 + \int_0^t \|u_t\|_2^2 \leq \|\nabla u_0\|_2^2 + 4c_8^2 \int_0^t \|u\|_2^2.$$

Moreover, by integrating and using Hölder inequality in time we have

$$(141) \quad \|u\|_2^2 \leq \left(\|u_0\|_2 + \int_0^t \|u_t\|_2 \right)^2 \leq 2\|u_0\|_2^2 + 2T_{\max} \int_0^t \|u_t\|_2^2.$$

Combining (140) and (141) we get

$$\int_0^t \|u_t\|_2^2 \leq \|\nabla u_0\|_2^2 + 8T_{\max} c_8^2 \left(\|u_0\|_2^2 + \int_0^t ds \int_0^s \|u_t(\tau)\|_2^2 d\tau \right).$$

By Gronwall inequality we then get that $\int_0^t \|u_t\|_2^2$ is bounded up to T_{\max} . By (141) we consequently get that also $\|u\|_2$ is bounded. Hence, by (140) also $\|\nabla u\|_2$ is bounded. So we contradict (8) and conclude the proof. \square

APPENDIX C. PROOF OF LEMMA 1

At first we denote $H = L^2(\Omega)$, $V = H_{\Gamma_0}^1(\Omega)$ and $W = L^m(\Gamma_1)$. Since V is dense in H , using [53, Theorem 2.1] and (24), (25), we obtain that

$$(142) \quad u \in C_w([0, T]; V).$$

The key point is to show that the energy identity holds. With this aim and fixed $0 \leq s \leq t \leq T$, we set θ_0 to be the characteristic function of the interval $[s, t]$. For small $\delta > 0$, let $\theta(\tau) = \theta_\delta(\tau)$ be 1 for $\tau \in [s + \delta, t - \delta]$, zero for $\tau \notin (s, t)$ and linear in the intervals $[s, s + \delta]$ and $[t - \delta, t]$. Next let η_ε be a standard mollifying sequence, that is, $\eta = \eta_\varepsilon \in C^\infty(\mathbb{R})$, $\text{supp } \eta_\varepsilon \subset (-\varepsilon, \varepsilon)$, $\int_{-\infty}^{\infty} \eta_\varepsilon = 1$, η_ε even and nonnegative, and $\eta_\varepsilon = \varepsilon^{-1} \eta_1(\tau/\varepsilon)$. Let $*$ denote time convolution. We approximate u , extended as zero outside $[0, T]$, with $v = \eta * (\theta u) \in C_c^\infty(\mathbb{R}; V)$. Then

$$(143) \quad 0 = \int_{-\infty}^{+\infty} \frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2 = \int_{-\infty}^{+\infty} (\nabla v, \nabla v_t).$$

Using standard convolution properties and the Leibnitz rule, we see that $v_t = \eta * (\theta' u) + \eta * (\theta u_t)$ in H , so that $\eta * (\theta u_t) \in C_c^\infty(\mathbb{R}; V)$. Then, by (143),

$$(144) \quad 0 = \int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \eta * (\theta' \nabla u)) + \int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \nabla (\eta * (\theta u_t))).$$

Using (25) and the fact that $u_t \in L^m((0, T) \times \partial\Omega)$ we can take $\phi = \eta * \eta * (\theta u_t)$ in (26). Then, multiplying by θ , integrating from $-\infty$ to ∞ and using standard properties of convolution, we can evaluate the second term in (144) in the following way:

$$(145) \quad \begin{aligned} \int_{-\infty}^{+\infty} (\eta * (\theta \nabla u), \nabla (\eta * (\theta u_t))) &= \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta \zeta) \eta * (\theta u_t) \\ &\quad + \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta g) \eta * (\theta u_t) - \int_{-\infty}^{+\infty} \|\eta * (\theta u_t)\|_2^2 \end{aligned}$$

Combining (144) and (145), and recalling that $\theta = \theta_\delta$, we obtain the first approximate energy identity

$$(146) \quad \begin{aligned} 0 &= \int_{-\infty}^{+\infty} (\eta * (\theta_\delta \nabla u), \eta * (\theta'_\delta \nabla u)) - \int_{-\infty}^{+\infty} \|\eta * (\theta_\delta u_t)\|_2^2 \\ &\quad + \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta_\delta \zeta) \eta * (\theta_\delta u_t) + \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta_\delta g) \eta * (\theta_\delta u_t) \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now we examine each term in (146) separately as $\delta \rightarrow 0$ and ε (i.e. η) is fixed. Since $\theta_\delta \rightarrow \theta_0$ a.e. and

$$\begin{aligned} \|\eta * (\theta_\delta \zeta)\|_{m', \Gamma_1} &\leq \|\zeta\|_{m', \Gamma_1}, & \|\eta * (\theta_\delta u_t)\|_{m', \Gamma_1} &\leq \|u_t\|_{m', \Gamma_1} \\ \|\eta * (\theta_\delta g)\|_2 &\leq \|g\|_2, & \|\eta * (\theta_\delta u_t)\|_2 &\leq \|u_t\|_2, \end{aligned}$$

using (22), (25) and Lebesgue Dominated Convergence Theorem we get the convergences

$$\begin{aligned}
 -I_2 &\rightarrow \int_{-\infty}^{+\infty} \|\eta * (\theta_0 u_t)\|_2^2 \\
 I_3 &\rightarrow \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta * (\theta_0 \zeta) \eta * (\theta_0 u_t) \\
 I_4 &\rightarrow \int_{-\infty}^{+\infty} \int_{\Omega} \eta * (\theta_0 g) \eta * (\theta_0 u_t).
 \end{aligned}
 \tag{147}$$

Next we decompose the term I_1 as

$$\begin{aligned}
 I_1 &= \int_{-\infty}^{+\infty} (\eta * (\theta_0 \nabla u), \eta * (\theta'_\delta \nabla u)) + \int_{-\infty}^{+\infty} (\eta * [(\theta_\delta - \theta_0) \nabla u], \eta * (\theta'_\delta \nabla u)) \\
 &:= I_5 + I_6
 \end{aligned}$$

Since $\theta_\delta \rightarrow \theta_0$ in $L^1(\mathbb{R})$, by (24) we have that $\eta * [(\theta_\delta - \theta_0) \nabla u] \rightarrow 0$ strongly in $L^\infty(0, T; H)$. Moreover, by (24),

$$\begin{aligned}
 \|\eta * (\theta'_\delta \nabla u)\|_{L^1(0, T; H)} &\leq \|\theta'_\delta\|_{L^1(\mathbb{R})} \|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0, T; H)} \\
 &\leq 2 \|\eta\|_{L^\infty(\mathbb{R})} \|\nabla u\|_{L^\infty(0, T; H)},
 \end{aligned}$$

so that $I_6 \rightarrow 0$ as $\delta \rightarrow 0$. Next we note that, by the properties of convolution and the specific form of θ_δ ,

$$\begin{aligned}
 I_5 &= \int_{-\infty}^{+\infty} \theta'_\delta (\eta * \eta * (\theta_0 \nabla u), \nabla u) \\
 &= \frac{1}{\delta} \int_s^{s+\delta} (\eta * \eta * (\theta_0 \nabla u), \nabla u) - \frac{1}{\delta} \int_{t-\delta}^t (\eta * \eta * (\theta_0 \nabla u), \nabla u).
 \end{aligned}$$

By (142), the function $(\eta * \eta * (\theta_0 \nabla u), \nabla u)$ is continuous, so

$$I_5 \rightarrow (\eta * \eta * (\theta_0 \nabla u)(s), \nabla u(s)) - (\eta * \eta * (\theta_0 \nabla u)(t), \nabla u(t)) \quad \text{as } \delta \rightarrow 0.
 \tag{148}$$

Combining the convergences (147)-(148) with (146), recalling that $\eta = \eta_\varepsilon$ and letting $\rho_\varepsilon = \eta_\varepsilon * \eta_\varepsilon$, we obtain the second approximate energy identity

$$\begin{aligned}
 (\rho_\varepsilon * (\theta_0 \nabla u), \nabla u)|_s^t &= \int_{-\infty}^{+\infty} \int_{\Gamma_1} \eta_\varepsilon * (\theta_0 \zeta) \eta_\varepsilon * (\theta_0 u_t) \\
 &\quad + \int_{-\infty}^{+\infty} \int_{\Omega} \eta_\varepsilon * (\theta_0 g) \eta_\varepsilon * (\theta_0 u_t) - \int_{-\infty}^{+\infty} \|\eta_\varepsilon * (\theta_0 u_t)\|_2^2.
 \end{aligned}
 \tag{149}$$

Now we consider the convergence of the two sides of (149) as $\varepsilon \rightarrow 0$. By standard arguments, using (25) and the fact that $u_t \in L^m((0, T) \times \Omega)$ we get that $\eta_\varepsilon * (\theta_0 u_t) \rightarrow \theta_0 u_t$ strongly in $L^m((0, T) \times \Gamma_1)$ and in $L^2((0, T) \times \Omega)$. Hence, using (22) and remembering that $g \in L^2((0, T) \times \Omega)$, the right-hand side of (149) goes to

$$\begin{aligned}
 &\int_{-\infty}^{+\infty} \int_{\Gamma_1} \theta_0^2 \zeta u_t + \int_{-\infty}^{+\infty} \int_{\Omega} \theta_0^2 g u_t - \int_{-\infty}^{+\infty} \|\theta_0 u_t\|_2^2 \\
 &= \int_s^t \int_{\Gamma_1} \zeta u_t + \int_s^t \int_{\Omega} g u_t - \int_s^t \|u_t\|_2^2.
 \end{aligned}$$

Concerning the left-hand side of (149), we note that $\text{supp } \rho_\varepsilon \subset (-2\varepsilon, 2\varepsilon)$, $0 \leq \rho_\varepsilon = O(\varepsilon^{-1})$ and $\int_0^{+\infty} \rho_\varepsilon = \int_{-\infty}^0 \rho_\varepsilon = \frac{1}{2} \int_{-\infty}^{+\infty} \rho_\varepsilon = \frac{1}{2}$. Therefore, for sufficiently small ε ,

$$(\rho_\varepsilon * (\theta_0 \nabla u)(t), \nabla u(t)) - \frac{1}{2} \|\nabla u(t)\|_2^2 = \int_0^{+\infty} \rho_\varepsilon(\tau) (\nabla u(t-\tau) - \nabla u(t), \nabla u(t)) d\tau.$$

Since, by (142), $\tau \mapsto (\nabla u(t-\tau) \nabla u(t), \nabla u(t))$ is continuous and vanishes when $\tau = 0$, we conclude that, as $\varepsilon \rightarrow 0$, $(\rho_\varepsilon * (\theta_0 \nabla u)(t), \nabla u(t)) \rightarrow \frac{1}{2} \|\nabla u(t)\|_2^2$. The same result, of course, continues to hold when t is replaced by s . Then we can pass to the limit in (149) and conclude the proof of (28). To show that (27) holds, we note that, by (28), it follows that $t \mapsto \|\nabla u(t)\|_2^2$ is continuous. Now we fix t in $[0, T]$ and let $t_k \rightarrow t$. Using (142), we have $\|u(t_k) - u(t)\|_V^2 = \|u(t_k)\|_V^2 + \|u(t)\|_V^2 - 2(u(t_k), u(t))_V \rightarrow 0$ as $k \rightarrow \infty$, concluding the proof.

REFERENCES

- [1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
- [2] H. Amann. Parabolic evolution equations and nonlinear boundary conditions. *J. Differential Equations*, 72(2):201–269, 1988.
- [3] J.-P. Aubin. Un théorème de compacité. *C. R. Acad. Sci. Paris*, 256:5042–5044, 1963.
- [4] G. Autuori and P. Pucci. Kirchhoff systems with nonlinear source and boundary damping terms. *Commun. Pure Appl. Anal.*, 9(5):1161–1188, 2010.
- [5] G. Autuori, P. Pucci, and M. C. Salvatori. Global nonexistence for nonlinear Kirchhoff systems. *Arch. Ration. Mech. Anal.*, 196(2):489–516, 2010.
- [6] I. Bejenaru, J. I. Díaz, and I. I. Vrabie. An abstract approximate controllability result and applications to elliptic and parabolic systems with dynamic boundary conditions. *Electron. J. Differential Equations*, pages No. 50, 19 pp. (electronic), 2001.
- [7] L. Bociu and I. Lasiecka. Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping. *Discrete Contin. Dyn. Syst.*, 22(4):835–860, 2008.
- [8] L. Bociu and I. Lasiecka. Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping. *J. Differential Equations*, 249(3):654–683, 2010.
- [9] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [10] H. Brezis and T. Cazenave. Unpublished book.
- [11] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa. Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differential Equations*, 1(1):73–90, 1996.
- [12] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and I. Lasiecka. Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping—source interaction. *J. Differential Equations*, 236(2):407–459, 2007.
- [13] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and P. Martinez. Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. *J. Differential Equations*, 203(1):119–158, 2004.
- [14] I. Chueshov, M. Eller, and I. Lasiecka. On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation. *Comm. Partial Differential Equations*, 27(9-10):1901–1951, 2002.
- [15] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [16] P. Colli. On some doubly nonlinear evolution equations in Banach spaces. *Japan J. Indust. Appl. Math.*, 9(2):181–203, 1992.
- [17] J. Ding and B.-Z. Guo. Blow-up and global existence for nonlinear parabolic equations with Neumann boundary conditions. *Comput. Math. Appl.*, 60(3):670–679, 2010.
- [18] J. Escher. Global existence and nonexistence for semilinear parabolic systems with nonlinear boundary conditions. *Math. Ann.*, 284(2):285–305, 1989.

- [19] J. Escher. Quasilinear parabolic systems with dynamical boundary conditions. *Comm. Partial Differential Equations*, 18(7-8):1309–1364, 1993.
- [20] J. Escher. On the qualitative behaviour of some semilinear parabolic problems. *Differential Integral Equations*, 8(2):247–267, 1995.
- [21] Z.-H. Fan and C.-K. Zhong. Attractors for parabolic equations with dynamic boundary conditions. *Nonlinear Anal.*, 68(6):1723–1732, 2008.
- [22] V. A. Galaktionov and J. L. Vázquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dyn. Syst.*, 8(2):399–433, 2002. Current developments in partial differential equations (Temuco, 1999).
- [23] V. Georgiev and G. Todorova. Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Differential Equations*, 109(2):295–308, 1994.
- [24] S. Gerbi and B. Said-Houari. Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions. *Adv. Differential Equations*, 13(11-12):1051–1074, 2008.
- [25] G. Gilardi and U. Stefanelli. Existence for a doubly nonlinear Volterra equation. *J. Math. Anal. Appl.*, 333(2):839–862, 2007.
- [26] M. Grobbelaar-van Dalsen. Semilinear evolution equations and fractional powers of a closed pair of operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 105:101–115, 1987.
- [27] T. Hintermann. Evolution equations with dynamic boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A*, 113(1-2):43–60, 1989.
- [28] K. Ishige and H. Yagisita. Blow-up problems for a semilinear heat equation with large diffusion. *J. Differential Equations*, 212(1):114–128, 2005.
- [29] M. Jazar and R. Kiwan. Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(2):215–218, 2008.
- [30] M. Kirane. Blow-up for some equations with semilinear dynamical boundary conditions of parabolic and hyperbolic type. *Hokkaido Math. J.*, 21(2):221–229, 1992.
- [31] I. Lasiecka. Stabilization of hyperbolic and parabolic systems with nonlinearly perturbed boundary conditions. *J. Differential Equations*, 75(1):53–87, 1988.
- [32] H. A. Levine. Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. *Arch. Rational Mech. Anal.*, 51:371–386, 1973.
- [33] H. A. Levine. The role of critical exponents in blowup theorems. *SIAM Rev.*, 32(2):262–288, 1990.
- [34] H. A. Levine, S. R. Park, and J. Serrin. Global existence and nonexistence theorems for quasilinear evolution equations of formally parabolic type. *J. Differential Equations*, 142(1):212–229, 1998.
- [35] H. A. Levine and L. E. Payne. Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time. *J. Differential Equations*, 16:319–334, 1974.
- [36] H. A. Levine and L. E. Payne. Some nonexistence theorems for initial-boundary value problems with nonlinear boundary constraints. *Proc. Amer. Math. Soc.*, 46:277–284, 1974.
- [37] H. A. Levine and J. Serrin. Global nonexistence theorems for quasilinear evolution equations with dissipation. *Arch. Rational Mech. Anal.*, 137(4):341–361, 1997.
- [38] H. A. Levine and R. A. Smith. A potential well theory for the heat equation with a nonlinear boundary condition. *Math. Methods Appl. Sci.*, 9(2):127–136, 1987.
- [39] H. A. Levine and R. A. Smith. A potential well theory for the wave equation with a nonlinear boundary condition. *J. Reine Angew. Math.*, 374:1–23, 1987.
- [40] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
- [41] J.-L. Lions and W. A. Strauss. Some non-linear evolution equations. *Bull. Soc. Math. France*, 93:43–96, 1965.
- [42] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.
- [43] E. Maitre and P. Witomski. A pseudo-monotonicity adapted to doubly nonlinear elliptic-parabolic equations. *Nonlinear Anal.*, 50(2, Ser. A: Theory Methods):223–250, 2002.
- [44] M. Marcus and V. J. Mizel. Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.*, 45:294–320, 1972.
- [45] N. Mizoguchi. Blowup rate of solutions for a semilinear heat equation with the Neumann boundary condition. *J. Differential Equations*, 193(1):212–238, 2003.

- [46] L. E. Payne and P. W. Schaefer. Blow-up in parabolic problems under Robin boundary conditions. *Appl. Anal.*, 87(6):699–707, 2008.
- [47] L. E. Payne and P. W. Schaefer. Blow-up phenomena for some nonlinear parabolic systems. *Int. J. Pure Appl. Math.*, 48(2):193–202, 2008.
- [48] L. E. Payne and J. C. Song. Lower bounds for blow-up time in a nonlinear parabolic problem. *J. Math. Anal. Appl.*, 354(1):394–396, 2009.
- [49] P. Pucci and J. Serrin. Global nonexistence for abstract evolution equations with positive initial energy. *J. Differential Equations*, 150(1):203–214, 1998.
- [50] G. Schimperna, A. Segatti, and U. Stefanelli. Well-posedness and long-time behavior for a class of doubly nonlinear equations. *Discrete Contin. Dyn. Syst.*, 18(1):15–38, 2007.
- [51] J. Serrin, G. Todorova, and E. Vitillaro. Existence for a nonlinear wave equation with damping and source terms. *Differential Integral Equations*, 16(1):13–50, 2003.
- [52] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.
- [53] W. A. Strauss. On continuity of functions with values in various Banach spaces. *Pacific J. Math.*, 19:543–551, 1966.
- [54] M. E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.
- [55] S. V. Uspenskiĭ. An imbedding theorem for S. L. Sobolev’s classes of fractional order $W_{p,r}$. *Soviet Math. Dokl.*, 1:132–133, 1960.
- [56] E. Vitillaro. Some new results on global nonexistence and blow-up for evolution problems with positive initial energy. *Rend. Istit. Mat. Univ. Trieste*, 31(suppl. 2):245–275, 2000. Workshop on Blow-up and Global Existence of Solutions for Parabolic and Hyperbolic Problems (Trieste, 1999).
- [57] E. Vitillaro. Global existence for the wave equation with nonlinear boundary damping and source terms. *J. Differential Equations*, 186(1):259–298, 2002.
- [58] E. Vitillaro. A potential well theory for the wave equation with nonlinear source and boundary damping terms. *Glasg. Math. J.*, 44(3):375–395, 2002.
- [59] E. Vitillaro. Global existence for the heat equation with nonlinear dynamical boundary condition. *Proc. Roy. Soc. Edinburgh Sect. A*, 135:1–33, 2005.
- [60] E. Vitillaro. On the Laplace equation with non-linear dynamical boundary conditions. *Proc. London Math. Soc. (3)*, 93(2):418–446, 2006.
- [61] J. von Below and G. Pincet Mailly. Blow up for reaction diffusion equations under dynamical boundary conditions. *Comm. Partial Differential Equations*, 28(1-2):223–247, 2003.
- [62] J. von Below and G. Pincet Mailly. Blow up for some nonlinear parabolic problems with convection under dynamical boundary conditions. *Discrete Contin. Dyn. Syst.*, (Dynamical Systems and Differential Equations. Proceedings of the 6th AIMS International Conference, suppl.):1031–1041, 2007.
- [63] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

(E. Vitillaro) DIPARTIMENTO DI MATEMATICA ED INFORMATICA, UNIVERSITÀ DI PERUGIA, VIA VANVITELLI, 1 06123 PERUGIA ITALY

E-mail address: enzo@dmf.unipg.it

(A. Fiscella) DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI UDINE, VIA DELLE SCIENZE, 206 33100 UDINE ITALY

E-mail address: alessio.fiscella@uniud.it